Shared Auxiliary Storage in a Machine-Verified 833-Qubit secp256k1 ECDLP Program

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Abstract

We reduce the fixed allocation of a machine-verified secp256k1 elliptic-curve discrete-logarithm program from 834 to 833 logical qubits. One allocated qubit serves as the zero-factor tag during totalized field division and multiplication and as the equality temporary during exceptional-input tagging and erasure and point negation. The two roles occur in disjoint program stages, and each stage restores the qubit before the next role begins. Lean 4 checks the revised point translation, two 256-step scalar loops, analytic selected-recovery probability bound, complete-program well-formedness, and exact allocation. The placement change preserves gate kinds and multiplicities, so every execution path retains the checked count of 781,290,954,752 compiled reversible Toffoli gates. The 833 equality states the program's fixed allocated width and gives a peak-live upper bound. It does not establish a minimum possible width.

1 Complete-program result

The revised program implements the complete double-scalar computation used by the semiclassical Shor algorithm for the elliptic-curve discrete-logarithm problem on secp256k1 [3, 2]. For the standard generator G, let q denote its prime order. For a valid public point $Q = [d]G$ with $0 < d < q$, and all 256-bit source scalars a and b, its two scalar loops construct

$$[a]G + [b]Q = [a + db]G.$$

The point operation covers the identity, ordinary addition, inverse pairs, and doubling. Its arithmetic workspace returns to its specified input state on every measurement branch. Summation over the internal measurement records gives the analytic scalar-pair distribution used by the recovery proof.

Set $N = 2^{256}$, and for $0 \le t < q$ define the nearest Fourier bin

$$J_N(t) = \left\lfloor \frac{2Nt + q}{2q} \right\rfloor.$$

The notation $[x]_q$ denotes the least nonnegative residue of x modulo q. For every $0 < t < q$, the selected scalar pair is

$$\big(J_N(t), J_N([dt]_q)\big).$$

The checked decoder maps every selected pair to d, and the one-run probability of the selected family is at least

$$\frac{q-1}{q}\,\frac{2401}{14641}.$$

The independent-product theorem proves that 26 executions contain at least one selected outcome with probability greater than 99/100, and every selected outcome recovers d. The preceding report established these functional and probability results for the 834-qubit program [4]. This note checks the same results after the allocation change.

VQ is a Lean 4 framework for exact functional verification and logical resource analysis of quantum programs [5, 1]. It represents unitary circuits, reversible basis permutations, and hybrid programs with measurement, reset, classical storage, and feed-forward. Its exact amplitude semantics retain each measurement branch. Functional proofs relate generated programs to mathematical specifications, while resource functions calculate width and path-dependent gate counts from the same program syntax. The secp256k1 development uses Mathlib for finite sums, complex analysis, modular arithmetic, and elliptic-curve groups [6].

2 Shared auxiliary qubit

Sharing logical qubit 832 between the zero-factor and equality roles reduces the complete program's fixed allocation from 834 to 833 logical qubits. The 834-qubit allocation assigned separate logical qubits to the two temporary Boolean values [4]. Totalized field division and multiplication use a zero-factor tag to replace a zero factor by one during a reversible arithmetic stage. Point negation and exceptional-case correction use an equality temporary while computing and erasing comparison results.

The division operation preserves its denominator in the input register and restores the zero-factor tag. The multiplication operation preserves one factor in the input register and restores the same tag. These are the retained division and retained multiplication used below.

The revised allocation assigns both roles to logical qubit 832, using zero-based indexing. The program uses the shared qubit in this order:

1. use the equality temporary to compute the four exceptional-input tags, clearing the temporary after each comparison;
2. compute the zero-factor tag, perform retained division, and clear the tag;
3. compute the zero-factor tag, perform retained multiplication, and clear the tag;
4. use the equality temporary for controlled negation and clear it;
5. apply the exceptional-case correction, then use the equality temporary to erase the four exceptional tags.

No stage reads one role while the qubit stores the other. The arithmetic proofs establish that the zero-factor tag is clear after division and multiplication. The comparison proofs establish that the equality temporary is clear after input tagging, negation, and exceptional-tag erasure. The composed translation proof uses these restoration results at the role boundaries. The shared allocation therefore preserves the point-translation function and gate multiplicities while changing the target index of operations on the shared qubit.

Three 256-qubit field registers use 768 qubits. The recycled Fourier control uses one. Extensions, control and sign bits, length and shift words, and the shared arithmetic and selector pool in the packed binary-Euclidean state use $6+4+35+14=59$. Four exceptional-case tags and the shared auxiliary qubit complete the allocation:

$$768 + 1 + 59 + 4 + 1 = 833.$$

Lean evaluates this layout width and proves that every gate and measurement references a valid location. This well-formedness result covers the complete two-loop program for every compiled public-point value.

3 Preserved semantics and resources

The revised translation proof tracks the shared qubit through retained division, retained multiplication, controlled negation, and correction. Each component receives the clear temporary required by its specification and restores it. Composition yields the same total group translation and workspace state as the prior allocation. The scalar-loop, marginal, and selected-recovery theorems then apply to the revised operation list.

The Toffoli metric counts one compiled CCZ for each reversible controlled-controlled-X operation, which the compiler represents as a CCZ conjugated by Hadamard gates on its target. We call each counted CCZ a Toffoli-class operation. The allocation change alters indices and restoration premises while preserving gate kinds, multiplicities, and branch structure. The syntax-derived resource range remains the singleton

$$\{781{,}290{,}954{,}752\}.$$

Thus every execution path has that Toffoli-class count, and its expectation under any output distribution has the same value. Measurements, classically selected Pauli operations, and phase rotations contribute zero to this metric. The count describes compiled reversible Toffoli-class operations and excludes error correction, routing, and physical gate synthesis.

The program uses the fixed 1,620-round binary-Euclidean schedule analyzed in the preceding report. A separately proved optimized Euclidean schedule is absent from this generated program and contributes no reduction to this count.

The exact width theorem concerns the program's declared quantum allocation. Because the program allocates one fixed register array, 833 also bounds its peak live logical qubits. The theorem does not prove that every allocation for this algorithm requires 833 qubits. A minimality result would require a lower bound over an explicit class of implementations.

4 Formal evidence

Lean checks the source against the repository's pinned compiler and Mathlib revisions. The checked chain contains the shared layout, component placement and well-formedness, basis-state semantics for each changed stage, complete point translation, two-scalar semantics, scalar-pair marginal, selected-recovery bounds, complete-program well-formedness, exact width, and exact pathwise Toffoli count. The semantic, probability, and resource statements refer to the same generated operation list.

The proof dependency set includes propositional extensionality, classical choice, quotient soundness, and private axioms generated by Lean's native decision procedure for fixed finite layout and gate-list computations. The semantic and resource theorems describe the exact logical program model. A physical resource estimate would require a fault-tolerant architecture, routing, phase synthesis, code parameters, and factory allocation.

References

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