Verified Arithmetic Reductions for an 833-Qubit secp256k1 ECDLP Program

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Abstract

We reduce the verified Toffoli bound for an 833-logical-qubit secp256k1 discrete-logarithm program from 36,591,123,006 to 18,467,700,570 per execution, preserving one-run recovery probability greater than 99 percent. The reductions combine paired endpoint scans, direct modular-correction predicates, restructured carry propagation, shared controls for length writes, and measurement-based erasure of arithmetic and selector temporaries. Lean proofs establish the changed program's exact branch amplitudes, restored quantum workspace, pathwise and expected Toffoli bounds, and unchanged Fourier-output distribution. A lifetime proof reuses the final Fourier-output bit for arithmetic measurement results, keeping the mutable classical allocation at 768 bits. The reported minimum uses five-bit scalar windows and a 26-bit initial lookup, with a 4 GiB packed public table and at most 34,493,956,094 lookup CNOTs. We derive the component savings, state the conditions for coherent measurement erasure, and describe proof decompositions that keep measurement histories and generated circuits symbolic.

1 Complete-program result

The verified program uses 833 logical qubits and at most 18,467,700,570 Toffoli-class operations per execution, with secp256k1 private-scalar recovery probability greater than 99/100. The preceding report [1] established the same allocation and recovery guarantee at 36,591,123,006 Toffolis. This report covers arithmetic changes that save 18,123,422,436 operations within that construction. The formal results described here belong to the source snapshot in [2].

Let $E/\mathbb{F}_p$ be secp256k1, with $p = 2^{256} - 2^{32} - 977$, and let G be its standard generator of prime order q. The public point is $Q = [d]G$, with $0 < d < q$, in canonical encoding. Put $N = 2^{256}$. The program performs coherent double-scalar evaluation $(a, b) \mapsto [a]G + [b]Q$, followed by semiclassical Fourier sampling. Its initial public-point lookup replaces the first scalar's leading binary digit and m five-bit digits, where $0 \leq m \leq 25$ and the lookup address has $k = 1 + 5m$ bits. The remaining translations comprise one binary step and $102 - m$ five-bit steps. Each five-bit table selects from P through $32P$, where P is that digit's public addend. Public offsets in the initial table cancel the shifts introduced by these nonzero digit tables [1].

Theorem 1 (Measured-arithmetic program)**.** *For every such public point and integer $0 \leq m \leq 25$, the program allocates 833 quantum bits and 768 mutable classical bits. Starting with zero quantum and mutable classical storage and using exact Fourier phases, it restores the arithmetic and address workspace on every branch. Its bounded decoder returns d with probability greater than $99/100$. If T is the execution's Toffoli-class count, there is an integer $C(Q, m)$ such that every path has count*

$C(Q, m)$ and

$$\mathbb{E}[T] = C(Q, m) \leq B(m) := 187{,}384{,}574 + (102 - m)187{,}765{,}022 + 2^{1+5m} - 2. \tag{1}$$

The minimum of $B(m)$ on this interval occurs at $m = 5$, where $B(5) = 18{,}467{,}700{,}570$.

The formal phase representation contains the 2^{ℓ}-th roots of unity for an integer $\ell \geq 257$. This supplies the exact rotations assumed in the theorem. The expectation identity also holds for every basis input within the quantum allocation, with zero initial mutable storage. Recovery uses the initialized state specified above. The quantum program declares zero immutable classical input bits. Its public tables and decoder require separate classical storage.

The Toffoli metric counts one compiled CCZ for each reversible Toffoli. Measurements, classically selected Clifford corrections, resets, CNOTs, and Fourier rotations have zero weight in this metric. Their resource costs require separate accounting. The fixed quantum allocation gives a peak-live upper bound of 833 logical qubits.

Construction	Complete bound	Reduction
Previous report [1]	36,591,123,006	
Paired endpoint scans	30,362,650,686	6,228,472,320
Direct correction predicates	25,517,213,362	4,845,437,324
Structured controlled carry	23,881,446,362	1,635,767,000
Shared length-write controls	23,075,270,138	806,176,224
Coefficient carry propagation	21,282,552,218	1,792,717,920
Measured ownership cells	19,714,919,130	1,567,633,088
Measured remainder cells	19,057,017,690	657,901,440
Direct terminal masked bit	19,051,937,370	5,080,320
Measured paired prefixes	18,467,700,570	584,236,800

Table 1: Checked complete-program Toffoli upper bounds with the 26-bit initial lookup. Every row retains 833 logical qubits and one-run recovery above 99/100. The reduction is relative to the preceding row.

VQ is a Lean 4 framework for exact functional verification and formally certified logical-resource analysis of quantum algorithms for cryptography and cryptanalysis. Its semantics retain complex amplitudes, measurement branches, and classical feed-forward. For this program, proofs connect reversible arithmetic to complete point translations, connect those translations to the double-scalar Fourier distribution, and bound resources for the same generated operations. The development uses Mathlib for algebra, finite sums, and analytic recovery. The reported execution, recovery, and resource results pass Lean 4.34.0-rc2 with propositional extensionality, classical choice, and quotient soundness as their transitive axioms [2].

2 Reversible arithmetic reductions

The packed field arithmetic retains a fixed schedule of 1,620 Euclidean rounds. A forward *prepared traversal* comprises input normalization, length initialization, and these rounds. Its inverse reverses the rounds and preparation. A complete point translation uses four prepared traversals through its division and multiplication operations. With a 26-bit initial lookup, the program executes 98 point translations. A saving of s Toffolis per prepared traversal therefore saves $392s$ in the complete program. Terminal arithmetic, phase correction, and exceptional-point handling contribute outside the prepared traversals.

2.1 Paired endpoint scans

The remainder pass updates 259 work bits between coherently stored endpoints. The prior construction retained each endpoint's six high address bits and recomputed its three low-bit comparisons at each position. The replacement retains five high bits and traverses four low bits with shared prefixes. It interleaves the left endpoint toggle, carry cell, and right endpoint toggle in the original order. The two endpoint traversals use three prefix temporaries each, fitting the existing scratch allocation.

For one endpoint, the selector overhead for a full four-bit block is 30 Toffolis. The three-position final block costs nine. Preparing and reversing the paired five-bit high comparisons costs 36 per block and direction. There are 16 full blocks and one partial block. Including both carry directions gives

$$T_{\text{pass}} = 4(16 \cdot 30 + 9) + 2 \cdot 17 \cdot 36 + 7 \cdot 259 = 4{,}993. \tag{2}$$

The prior pass cost 9,897. Two passes per round therefore save $2 \cdot 1{,}620 \cdot 4{,}904 = 15{,}888{,}960$ per prepared traversal. The result completes the paired-scan proposal in the preceding report [1].

The prefix-sharing identity is $(c \wedge \neg b) \oplus c = c \wedge b$. It replaces separate erasure and preparation between sibling prefixes by one CNOT, following the prefix reuse of unary iteration [4]. The paired proof factors both endpoint computations through each carry cell under explicit preservation and commutation conditions. Induction over the pruned traversal proves the full action, leaf order, and scratch restoration, including the partial block and reversed order.

2.2 Direct modular-correction predicates

Reduction modulo p uses the gap $2^{32} + 977$ in chunks. Above the low chunk, the correction needs overflow from adding a carry to a zero constant and borrow from subtracting one. For $0 \le x < 2^w$ and $c \in \{0, 1\}$,

$$x + c \ge 2^w \iff c = 1 \text{ and } x = 2^w - 1, \qquad x - 1 < 0 \iff x = 0.$$

Including the enable bit gives conjunctions on $w+2$ and $w+1$ controls. Their reversible implementations cost $2w+1$ and $2w-1$ Toffolis and use the cleared constant register as temporary storage. The original add-copy-reverse probes each cost $2(10w + 1)$. The replacements preserve the chunk, controls, and arbitrary initial output flag and restore the borrowed register.

The resulting modular doubling and controlled modular addition cost 5,899 and 9,485 Toffolis. A 256-bit product costs 3,932,405, a reduction of 6,171,347. The translation's product occurrences save 49,370,776, and its modular additions save another 72,462. Their sum gives $49{,}443{,}238 \cdot 98 = 4{,}845{,}437{,}324$ at program level.

2.3 Controlled carry propagation

The controlled carry adder computes an unconditional Cuccaro majority chain [3] and applies the enable bit during the reverse sum updates. An enabled reverse cell first inverts its majority cell, forms source–carry parity in the source bit, toggles the target under that parity and the enable bit, and restores the source. A disabled cell inverts the majority cell. One controlled copy records the final carry. The count is

$$w + 2w + 1 = 3w + 1,$$

compared with $10w + 1$ for gatewise control. The intermediate construction used two controlled sum toggles and cost $4w + 1$. Parity sharing saves one Toffoli per cell at the cost of two CNOTs.

The cell identities and chain induction hold for arbitrary basis contents, including the incoming carry and output-carry bit. They preserve the source, enable, and existing spare bit. The replacement fits the same $2w+4$-qubit layout. Its integration reduces modular doubling to 2,721 Toffolis, controlled modular addition to 4,515, and the 256-bit product to 1,849,695. The translation saves 16,691,500, giving the complete-program reduction in Table 1.

2.4 Shared controls for length writes

A length writer XORs a public constant into a register under the conjunction of two quantum controls. If the constant has $h > 0$ set bits, the construction selects one target as a pivot. The circuit applies CNOTs from that pivot to every other selected target, toggles the pivot with one Toffoli, and repeats the CNOTs. For each other target, the pivot's initial value cancels and the common control conjunction remains. The construction preserves arbitrary initial target values and uses one Toffoli and $2(h-1)$ CNOTs. A zero constant emits no gates.

The *ownership update* exchanges the packed Euclidean work fields and updates the length fields that identify their active portions. Its upper and lower length writers traverse round-dependent windows. For round $1 \le s \le 1{,}620$, let $a_s = \lfloor s/4 \rfloor$ and $r_s = a_s \bmod 49$. The inherited coefficient bounds give

$$\begin{aligned} b_s &= \max\{1, 31\lfloor a_s/49\rfloor + 12\lfloor r_s/19\rfloor + 5\lfloor (r_s \bmod 19)/8\rfloor\}, \\ u_s &= 1 + \min\{258, a_s + 3\}, \qquad l_s = 258 - b_s. \end{aligned} \tag{3}$$

Here u_s and l_s are the upper and lower writer widths. For a nonempty window of width w, shared controls reduce the constant-write contribution from $9 + 2(w-1) - \mathrm{popcount}(w-1)$ to w. With $\Delta(w) = 7 + w - \mathrm{popcount}(w-1)$, the schedule saving is $4\sum_{s=1}^{1620}(\Delta(u_s) + \Delta(l_s)) = 2{,}056{,}572$. The full-width writers each fall from 6,258 to 5,730 Toffolis.

2.5 Boundary-selected coefficient addition

The coefficient pass adds and subtracts on a prefix selected by a stored boundary. Its former implementation controlled both carry directions and cost $11w - 4$ Toffolis per operation. The replacement computes the majority chain across the whole work field, then reverses the cells above the selected boundary. This restores the carry at the boundary before the active prefix is updated. The reverse traversal performs the enabled sum updates at and below the boundary. A sign-producing addition copies the boundary carry before those updates. An inactive outer control reverses the complete chain.

The reverse cell costs two Toffolis. A single boundary traversal gives $5w - 2$ without sign extraction, and the selected carry copy raises that count to $6w - 2$. At $w = 257$, subtraction and sign-producing addition cost 1,283 and 1,540, respectively, with 1,798 CNOTs each. The former pair cost $2 \cdot 2{,}823$, so the round saves 2,823 Toffolis. The prepared traversal falls to 50,309,352. The proof covers arbitrary source and carry contents, coherent boundary selection, inactive control, and restoration of routing scratch. The active boundary lies in the declared label interval.

3 Measurement-based erasure

Suppose a temporary t equals the conjunction of two preserved bits a and b. For arbitrary amplitudes and additional data z, the state before erasure has the form

$$\sum_{a,b,z} \alpha_{a,b,z} |a, b, z\rangle |ab\rangle.$$

Applying a Hadamard to t and measuring it produces outcome $r \in \{0,1\}$ with unnormalized data amplitude $2^{-1/2}(-1)^{rab}\alpha_{a,b,z}$. A CZ on a, b when $r = 1$ cancels the phase. An X on t when $r = 1$ restores t to zero. Each corrected branch is therefore

$$2^{-1/2} \sum_{a,b,z} \alpha_{a,b,z} |a, b, z\rangle |0\rangle. \tag{4}$$

This is measurement-based AND erasure [5]. It replaces one reversible erasure Toffoli with one measurement and Clifford corrections. The implementation restores zero by conditional X and uses zero explicit reset operations. Its validity requires the stored conjunction relation at the time of measurement.

3.1 Ownership zero relations

The ownership writers test masked zero relations while scanning the upper and lower active ranges. An internal cell computes a conjunction into a clean temporary, uses it to update the result, and erases it. Equation (4) reduces that cell from three Toffolis to two and adds one measurement. The surrounding operations preserve the conjunction controls until erasure.

Let H be the selector overhead of one range traversal. A width-w zero scan with its original terminal cell then costs $2H + 4(w-1) + 2$ Toffolis and $2(w-1)$ measurements. A complete length writer saves $4(w-1)$ Toffolis and adds the same number of measurements. At width 259, its count falls from 5,730 to 4,698, with 1,032 measurements.

For the widths in (3), the ownership update saves $8((u_s - 1) + (l_s - 1))$. The exact sum over the existing round-dependent windows is

$$\sum_{s=1}^{1620} 8((u_s - 1) + (l_s - 1)) = 3{,}999{,}064. \tag{5}$$

The prepared traversal therefore costs 46,310,288 Toffolis and 3,999,064 measurements in either arithmetic direction. Extending the replacement through every translation saves $392 \cdot 3{,}999{,}064 = 1{,}567{,}633{,}088$ per execution. The first complete integration used measured arithmetic in the five-bit steps and had bound 19,730,915,386. Replacing the leading binary translation saved the remaining 15,996,256, yielding the ownership row of Table 1.

3.2 Controlled remainder cells

The remainder carry cells contain a three-control NOT decomposed into three Toffolis with a clean temporary. Erasing the temporary through (4) gives a two-Toffoli majority cell and a three-Toffoli inverse-majority cell, each with one measurement. Both arithmetic directions preserve their original action and return the temporary to zero.

Each remainder pass visits the 259 positions in both carry directions. It saves $2 \cdot 259 = 518$ Toffolis, reducing (2) to 4,475, with 518 measurements. Two passes per round save 1,036. The prepared traversal then costs 44,631,968 Toffolis and 5,677,384 measurements, giving

$$1{,}036 \cdot 1{,}620 \cdot 4 \cdot 98 = 657{,}901{,}440$$

fewer Toffolis per complete execution.

3.3 Direct terminal masked bit

The terminal cell of each ownership zero scan toggles its result by $1 \oplus (rs)$, where r is the range bit and s is the source bit. An X followed by one Toffoli implements that action. The former cell computed and erased a temporary conjunction with two Toffolis. The direct construction saves eight Toffolis per round, or 12,960 per prepared traversal. Its complete saving is 5,080,320. Measurement counts remain unchanged, so subsequent formulas track saved Toffolis and added measurements separately.

3.4 Paired selector prefixes

An internal selector node retains either $c \wedge b$ or $c \wedge \neg b$, where c is its parent predicate and b is an endpoint bit. Its children must preserve c, b, and the stored predicate and restore their descendant scratch. After the negative child, a CNOT controlled by c changes the stored negative predicate to the positive one. Equation (4) then erases that conjunction after the positive child. A node ending in the negative polarity first applies the same CNOT. The proof covers both traversal directions and pruned children.

A full four-bit tree has seven active internal nodes above its final address bit. Erasing both endpoint predicates saves 14 Toffolis, reducing the paired routing cost from 60 to 46. The three-position tail saves six, reducing its cost from 18 to 12. The two carry directions save

$$2(16 \cdot 14 + 6) = 460$$

per remainder pass. The resulting pass costs 4,015 Toffolis and 978 measurements. Two passes per round give

$$920 \cdot 1{,}620 \cdot 4 \cdot 98 = 584{,}236{,}800 \tag{6}$$

saved Toffolis and the same number of added measurements.

This replacement requires zero selector scratch as well as a zero cell temporary. The proof establishes these conditions after high-endpoint preparation and at both remainder-operation boundaries. The encoded Euclidean state equations provide the conditions for reachable active states and for the terminal states used to pad the fixed schedule. Reversible cancellation identifies the corresponding inverse-entry states. Those results justify composition through the whole prepared traversal under its original public input hypotheses.

4 Arithmetic and Fourier composition

Division and multiplication preserve an operand while using forward and inverse prepared traversals and measured intermediate results. The new traversal proofs apply to positive canonical field operands and preserve every quantum field outside their placement. Zero-source selection temporarily replaces zero by one, performs the selected arithmetic, and restores the original zero case. Complete point translations retain the existing correction for inputs $\mathcal{O}$, P, $-P$, and $-2P$, where P is the selected public addend [1].

The retained intermediate measurements require classical positions 0–255 until phase correction. Division reverses its prepared traversal before that correction. Multiplication performs a forward traversal after product measurement and before phase correction. The new proofs retain the measured bits through these traversals and cancel the original product-dependent phase. Every additional erasure contributes only the common factor in (4).

Classical position 767 stores the erasure outcome. The first scalar loop writes its Fourier result to positions 256–511. The second writes positions 512–767 in increasing order. Its final arithmetic call precedes the Fourier measurement that writes position 767. Each erasure and prepared traversal preserves positions below 767. Complete division and multiplication preserve Fourier-result positions 256–766. The final Fourier step has the same complete branch list for either incoming value of that last position, because its final measurement overwrites the value before any subsequent use. This lifetime argument keeps the mutable classical allocation at 768 bits while preserving every phase-correction bit and earlier Fourier result.

Let $M(m)$ count internal measurements, excluding the 512 Fourier outputs. In the final construction,

$$M(m) = 2^{1+5m} - 2 + 28{,}671{,}648(103 - m). \tag{7}$$

For each Fourier-result pair, the branch-count theorem gives $2^{M(m)}$ internal histories. The execution theorem gives the same terminal quantum state on each history, multiplied by $2^{-M(m)/2}$. Thus summing their probabilities cancels the internal factor: $2^{M(m)}(2^{-M(m)/2})^2 = 1$.

The common scalar-pair distribution is

$$F_N(j,t) = \left| \frac{1}{N} \sum_{a=0}^{N-1} \exp\left(2\pi i a \left(\frac{j}{N} - \frac{t}{q}\right)\right) \right|^2, \tag{8}$$

$$\mu_d(j,h) = \frac{1}{q} \sum_{t=0}^{q-1} F_N(j,t) F_N(h,[dt]_q), \tag{9}$$

where $0 \le j,h < N$, $0 \le t < q$, and $[x]_q$ denotes the least nonnegative residue modulo q. The preceding report proves that its bounded decoder succeeds with probability greater than 99/100 under (9). The changed program has exactly this marginal, so the same analytic bound applies. The decoder checks cyclic neighbors of the observed Fourier labels within radius 128. Its proved work bounds remain 514 scalar-multiplication calls, 66,049 candidate comparisons, and one modular inversion.

5 Resource accounting

The final prepared traversal costs 43,128,608 Toffolis and 7,167,784 measurements in either direction. Its Toffoli saving from the 50,309,352-count reversible traversal is 7,180,744. The difference between that saving and its measurement count is 12,960, the direct terminal-cell reduction. The traversal uses zero explicit resets. A complete field division or multiplication costs 91,818,823 Toffolis and $2 \cdot 7{,}167{,}784 + 256 = 14{,}335{,}824$ measurements. The added 256 are the retained intermediate-result measurements.

A fixed-point translation costs 187,384,574 Toffolis and 28,671,648 measurements. A five-bit table-selected translation has an exact public-table-dependent Toffoli count bounded by 187,765,022 and the same measurement count. The initial lookup costs $2^k - 2$ Toffolis. Adding one fixed translation, $102-m$ table-selected translations, and the lookup proves (1). The classical alternatives introduced by measurement erasure contain zero Toffoli-class operations, so they preserve path independence. Normalization of the semantic branch probabilities gives the expected-count identity in Theorem 1.

The finite difference

$$B(m+1) - B(m) = -187{,}765{,}022 + 31 \cdot 2^{1+5m}$$

is negative through $m = 4$ and positive from $m = 5$ onward. It proves the minimum at $m = 5$ within the supported lookup family. The lower arithmetic cost changes the penalty for choosing a smaller prefix, while the initial lookup's CNOT and public-table bounds remain those of the preceding construction.

Prefix bits	Complete Toffoli bound	Lookup CNOT bound	Packed table
11	18,963,888,820	1,052,670	128 KiB
16	18,776,187,286	33,685,502	4 MiB
21	18,590,453,880	1,077,936,126	128 MiB
26	18,467,700,570	34,493,956,094	4 GiB
31	20,360,310,332	1,103,806,595,070	128 GiB

Table 2: The lookup-family tradeoff after measured prefix erasure. The lookup CNOT bound is $514 \cdot 2^k - 2$. Packed table storage is $64 \cdot 2^k$ bytes for 512-bit point codes.

The 21-bit choice increases the Toffoli bound by 122,753,310 and reduces the table from 4 GiB to 128 MiB. Table 2 counts the initial lookup's CNOTs and packed data. Complete-program CNOTs and generated circuit-representation storage require further accounting. The 768 mutable bits describe the quantum program's live classical register, while $M(m)$ describes the length of its internal measurement history.

6 Proof decomposition and remaining bounds

The formalization proves local full-state identities before composing the arithmetic. A measured component's statement specifies its input predicate, output state, restored temporaries, retained classical prefix, and branch-amplitude factor. Injective register placement preserves those statements. A component can therefore replace its reversible reference after the caller proves the input predicate. Forward and inverse arithmetic implementations have separate action proofs, including the intermediate states where erasure occurs.

The Euclidean schedule proof uses an abstract trace of reference states and proves each component's entry condition along that trace. Phase-specific equations supply clean scratch in active rounds, and terminal-state equations supply it after arithmetic has finished. This decomposition avoids expansion of the generated gate sequence when establishing a condition at an internal boundary. Structural count lemmas sum per-cell and per-block formulas, with exact finite arithmetic confined to the round-width sums.

The measurement-history proof treats the number of branches per step as a symbolic parameter and specializes it after the induction. Generic power identities establish history multiplication and amplitude cancellation. Scalar amplitudes use a fold with a proved composition equation. These decompositions reduce Lean's finite evaluation and repeated elaboration while retaining the complete branch semantics and exact counts.

The bounds assume exact logical gates and Fourier phases. Approximate rotation synthesis needs an error budget and a recovery analysis including that error. Physical error correction, routing, classical feed-forward latency, and gate factories need additional cost models. Measurement depth, complete CNOT counts, and classical decoder bit complexity also remain open. The stated minimum concerns the uniform bound in (1). Global Toffoli optimality and minimal quantum allocation require further results.

The remaining ownership selectors retain single endpoint prefixes, so the same erasure identity offers a further candidate. Its complete-program reduction requires scratch-entry proofs through the round and prepared schedule and composition of the resulting counts. Further carry restructuring and tighter public-table-dependent arithmetic bounds can be assessed within the established point and Fourier composition.

References

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