

An 833-Qubit secp256k1 Discrete-Logarithm Algorithm with Machine-Checked Correctness and Resource Bounds

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Abstract

We describe a quantum algorithm that recovers every nonzero secp256k1 private scalar with probability greater than 99 percent in one execution, using 833 logical qubits and at most 16,923,769,370 Toffoli-class operations. Lean proofs connect the generated hybrid program to exact group arithmetic, restored quantum workspace, its Fourier-output distribution, a bounded classical decoder, and pathwise and expected resource counts. The construction combines packed Euclidean arithmetic, five-bit scalar windows, complete point translations, an initial public lookup, and measurement with phase correction. The stated Toffoli bound uses a 26-bit initial lookup with a 4 GiB packed public table and at most 34,493,956,094 lookup CNOTs. Smaller tables give nearby Toffoli bounds. The decoder uses 514 scalar multiplications, at most 66,049 comparisons, and at most one modular inversion. We give the algorithm, the recovery argument, the allocation and cost derivations, and a map to the checked statements. We compare published low-qubit and low-Toffoli constructions, distinguishing universal formal proofs, circuit tests, statistical guarantees, and cryptographic attestations of tests. The result concerns ideal logical operations with exact Fourier phases. Physical error correction, phase synthesis, and complete depth and CNOT accounting require further analysis.

1 Problem and complete-program theorem

The algorithm recovers every valid nonzero secp256k1 private scalar with probability greater than 99/100 in one execution, within 833 allocated logical qubits and a uniform bound of 16,923,769,370 Toffoli-class operations. This report describes the completed construction in VQ, a Lean framework for quantum-algorithm verification. At revision `c46c289`, its execution, recovery, and resource proofs passed Lean 4.34.0-rc2 on 10 September 2026 [1]. Earlier reports describe intermediate optimizations [2, 3].

The curve is

$$E/\mathbb{F}_p : \quad y^2 = x^3 + 7, \quad p = 2^{256} - 2^{32} - 977.$$

Its standard generator G has prime order q , specified with G in Appendix A. The input is a canonically encoded point $Q = [d]G$ with $0 < d < q$. Put $N = 2^{256}$ and let $[z]_q$ denote the least nonnegative residue of an integer z modulo q . An affine point (x, y) uses the 512-bit code $x + 2^{256}y$ with $0 \leq x, y < p$. The code $(0, 0)$ represents $\mathcal{O}$. It lies off the affine curve, so these encodings are distinct [4, 1].

The program depends on the public point Q and a prefix parameter $m \in \{0, \dots, 25\}$. Define $k = 1 + 5m$. It constructs the double-scalar value $[a]G + [b]Q$ within two semiclassical Fourier

loops, each covering all 256 scalar bits. An initial lookup replaces the first k source bits of the first loop. The remaining coherent arithmetic uses one leading binary translation and $102 - m$ five-bit translations.

A phase level λ supplies exact roots of unity of order 2^λ for the program's phase operations.

Theorem 1 (Complete 833-qubit program). *For every $Q = [d]G$ as above and every $0 \leq m \leq 25$, the generated program has quantum allocation 833, mutable classical allocation 768, and zero declared immutable input bits. Starting from zero quantum and mutable classical storage, with exact Fourier phases at phase level at least 257, it has the following properties.*

1. *Every terminal branch has the double-scalar Fourier amplitudes specified in Section 5, with restored arithmetic and address workspace and preserved immutable input.*
2. *The bounded decoder in Section 6 returns d with probability greater than 99/100.*
3. *There is an exact integer count $C(Q, m)$ such that every execution path uses $C(Q, m)$ compiled CCZ operations and*

$$\mathbb{E}[T] = C(Q, m) \leq B(m) := 171,630,174 + (102 - m)172,010,622 + 2^{1+5m} - 2. \quad (1)$$

The minimum of this upper bound on the supported parameter range occurs at $m = 5$, with $B(5) = 16,923,769,370$.

The expectation equality holds under the program's semantic branch probabilities. Its resource proof also covers every in-width computational-basis input with zero initial mutable classical storage. The recovery theorem uses the all-zero initialization and valid-key hypotheses stated above. The allocated width, together with valid operation indices, bounds peak live quantum storage by 833.

One counted CCZ is equivalent to one Toffoli gate under Hadamard conjugation on its target. The metric assigns zero weight to CNOTs, measurements, resets, classically selected Clifford operations, and Fourier rotations. Section 7 gives the measured cleanup and lookup costs separately. The public tables and the classical decoder occupy storage outside the quantum program's 768-bit mutable register.

2 Scalar schedule and initial lookup

The construction uses Shor's discrete-logarithm reduction and the semiclassical Fourier transform [5, 6, 7]. The Fourier transform and measurement permit reuse of the quantum source positions once a digit's controlled point translation has finished. Recorded scalar-result bits determine later phase corrections. This leaves the point accumulator coherent throughout both scalar loops.

2.1 Nonzero digit tables

Write each scalar in the form

$$a = \epsilon_a 2^{255} + \sum_{j=0}^{50} r_{a,j} 32^j, \quad b = \epsilon_b 2^{255} + \sum_{j=0}^{50} r_{b,j} 32^j,$$

where $\epsilon_a, \epsilon_b \in \{0, 1\}$ and each $r_{a,j}, r_{b,j}$ belongs to $\{0, \dots, 31\}$. For a five-bit digit and public base point P , address r selects $[r + 1]P$. All 32 selected addends are finite and nonzero because their order is the prime $q > 32$. The table's coordinates enter one shared arithmetic computation.

Let

$$s = \sum_{j=0}^{50} 32^j, \quad P_0 = -[s](G + Q).$$

The accumulator starts at P_0 . The first loop's leading binary digit adds $\epsilon_a[2^{255}]G$. Its other digits add $[r_{a,j} + 1][32^j]G$. The second loop has the corresponding schedule for Q . The added constants sum to $[s](G + Q)$, cancelling P_0 , and the final point is $[a]G + [b]Q$. The binary digit has ordinary zero-or-one control and introduces no extra shift.

The bases are processed from the highest source weight to the lowest. At each digit, Hadamards prepare the address, the selected translation updates the accumulator, and the semiclassical Fourier block applies the phases determined by prior result bits, measures the next result bits, and clears the address. The proof retains both the source label and the recorded result prefix. Its invariant fixes the bit order and the sign of the Fourier phase, including interference between distinct source labels that encode the same group point.

2.2 Prefix preparation

For $k = 1 + 5m$, define

$$D_m = \sum_{j=51-m}^{50} 32^j,$$

with $D_0 = 0$. The initial public table contains

$$\mathcal{T}_{Q,m}[v] = P_0 + [v2^{256-k} + D_m]G, \quad 0 \leq v < 2^k. \quad (2)$$

The term D_m accounts for the m nonzero digit-table shifts consumed by this prefix. Every entry uses the canonical point encoding, including the identity when it occurs. Preparing a uniform k -bit address and looking up this table therefore replaces the leading binary step and the first m five-bit steps of the G loop.

The lookup traverses binary address prefixes. A prefix conjunction controls the public coordinate-bit toggles at each leaf. Each temporary conjunction is erased by measurement and phase correction. The traversal costs $2^k - 2$ Toffolis and fits in $512 + 2k$ quantum positions. The supported range has $k \leq 126$, within the proved placement condition $k \leq 128$. After the prefix Fourier block measures and clears the address, the accumulator enters the ordinary 833-position arithmetic layout.

The complete quantum procedure consists of this lookup and its Fourier block, $51 - m$ five-bit steps with bases $[32^{50-m}]G$ through G , one binary step with $[2^{255}]Q$, and 51 five-bit steps with bases $[32^{50}]Q$ through Q . Classical decoding follows the two measured 256-bit results. Preparation of the public tables uses only public group operations.

3 Packed field arithmetic

Field division and multiplication use a fixed reversible Euclidean schedule, with packed remainders and coefficients. Length registers locate the occupied parts of two shared work words. Reversible comparisons and arithmetic act on the selected intervals, preserving the remaining bits. The approach develops the register-sharing construction of Proos–Zalka and Luo et al. [6, 19]. The generated operations and their correctness statements are identified in Appendix C.

3.1 Euclidean computation and fixed termination

For a nonzero reduced input x , preparation computes $x_0 = \min(x, p - x)$ and retains the choice needed to recover the inverse of x . Since $0 < x_0 < 2^{255}$, its nonzero remainder lengths fit in eight bits. The mathematical Euclidean recurrence is

$$r_{-1} = p, \quad r_0 = x_0, \quad a_i = \lfloor r_{i-1}/r_i \rfloor, \quad r_{i+1} = r_{i-1} - a_i r_i.$$

Signed coefficients satisfy $t_{-1} = 0$, $t_0 = 1$, and $t_{i+1} = t_{i-1} - a_i t_i$, so $r_i \equiv t_i x_0 \pmod{p}$. At the last nonzero remainder, $r_i = 1$, giving the inverse coefficient.

The circuit implements this recurrence through a four-phase binary division schedule. It shifts and subtracts remainder intervals, updates the coefficient prefixes and quotient length, and changes the phase and ownership metadata. Here *ownership* identifies which packed interval belongs to which mathematical quantity. The packed-state invariant relates every occupied interval, length, phase bit, and sign to the mathematical state. Per-round interval bounds restrict the selector scans while covering every reachable state.

The schedule length follows a worst-case bound on the Euclidean quotient weight

$$W(p, x_0) = \sum_i (\lfloor \log_2 a_i \rfloor + 1).$$

The checked bound is $W(p, x_0) \leq 405$ for the preprocessed 256-bit input. Four circuit phases per weight unit give 1,620 rounds. The proof uses the finite minimum of four linear forms in the two remainders and its growth under a Euclidean step, followed by an exact numerical growth certificate. The formal results also cover the terminal padding, including its stored displacement, and the final inverse extraction. This makes the execution length independent of the quantum input.

Terminal preparation puts the reduced inverse coefficient z into the second work word, with the sign corrected so that $xz \equiv 1 \pmod{p}$. The terminal product uses this coefficient and the retained operand. Reversing terminal preparation and the Euclidean schedule restores the input and arithmetic workspace. The field callers handle a zero factor explicitly.

3.2 Total division, multiplication, and erasure

Define $\sigma(0) = 1$ and $\sigma(x) = x$ for $x \neq 0$ in $\mathbb{F}_p$. The in-place arithmetic interfaces are

$$(x, y, 0) \mapsto (x, y/\sigma(x), 0), \quad (x, y, 0) \mapsto (x, \sigma(x)y, 0). \quad (3)$$

A zero-factor predicate changes a zero factor to one before the nonzero arithmetic and restores its encoding afterward. The arithmetic retains this predicate until its cleanup conditions hold. Both maps are reversible on all reduced coordinate pairs.

The division caller first moves the numerator into the additional field. It prepares the denominator's Euclidean state, corrects the inverse coefficient, clears the stored modulus from the first work word, and forms the quotient there. It then measures the obsolete numerator in the X basis, moves the quotient into the released field, restores the terminal encoding, and reverses the Euclidean computation. From the retained denominator and quotient, it reconstructs the obsolete numerator for phase correction and clears the reconstruction. The multiplication caller has the corresponding proved in-place action and restored workspace.

The phase argument applies to an arbitrary coherent input

$$\sum_z \alpha_z |z\rangle |f(z)\rangle.$$

Measuring the h -bit second register in the X basis with result r gives the subnormalized remaining state

$$2^{-h/2} \sum_z \alpha_z (-1)^{r \cdot f(z)} |z\rangle.$$

Here $r \cdot f(z)$ is the binary inner product. Reconstructing $f(z)$ from the retained data and applying the same sign cancels this phase. Reset returns the measured storage to zero. Each outcome then has the same corrected amplitude factor $2^{-h/2}$. The cleanup proof establishes the reconstruction and preservation conditions at each use [8, 1].

The same identity erases one-bit conjunctions inside Euclidean arithmetic and prefix selectors. For $f(u, v) = u \wedge v$, a classically selected CZ supplies the phase correction after the X -basis measurement. The current program uses this method in remainder cells, paired interval scans, ownership updates, and high-endpoint selectors. These replacements save Toffolis and increase the measurement count reported in Section 7.

4 Complete point translation and allocation

The translation circuit implements $P \mapsto P + S$ for every valid accumulator, including exceptional affine cases and the identity. For a five-bit address, S is the selected nonzero public point. The circuit preserves that address throughout the translation. Its arithmetic starts from reduced coordinates, and every field operation returns reduced coordinates.

4.1 An invertible affine map

Fix $S = (A, B)$. The raw arithmetic on any $(x, y) \in \mathbb{F}_p^2$ is

$$\begin{aligned} \delta &= x - A, & \lambda &= (y - B)/\sigma(\delta), & u &= \delta - \lambda^2 + 3A, \\ X &= A - u, & Y &= \lambda\sigma(u) - B. \end{aligned} \tag{4}$$

It uses one in-place division, one subtraction of a square, one in-place multiplication, and selected constant-coordinate updates. Its inverse on all coordinate pairs is

$$\begin{aligned} u &= A - X, & \lambda &= (Y + B)/\sigma(u), & \delta &= u + \lambda^2 - 3A, \\ x &= \delta + A, & y &= \lambda\sigma(\delta) + B. \end{aligned} \tag{5}$$

Every denominator is nonzero by the definition of σ . Substitution proves that the two maps are inverse.

For an ordinary curve input, the slope is $(y - B)/(x - A)$, and $X = \lambda^2 - x - A$. Since $u = A - X$, the second output coordinate is the affine sum formula $Y = \lambda(A - X) - B$. The input exceptions are

$$\mathcal{E}_S = \{\mathcal{O}, S, -S, -2S\}.$$

The first three cover the identity and vanishing first denominator. For a remaining valid input, the second factor u can vanish only when the group sum has the same x -coordinate as S . The sum is then S or $-S$, giving the input $\mathcal{O}$ or $-2S$. Thus both divisions or multiplications use their ordinary nonzero factor on every input outside $\mathcal{E}_S$.

The raw outputs of the four exceptional inputs are distinct by injectivity of (4). Comparisons with these public outputs identify the exceptional case after the arithmetic. With that case recorded, selected coordinate updates perform

$$\mathcal{O} \mapsto S, \quad S \mapsto 2S, \quad -S \mapsto \mathcal{O}, \quad -2S \mapsto -S.$$

The four corrected outputs are distinct because S has prime order $q > 3$. Output comparisons erase the case data. Every ordinary valid input has the correct group output and cannot match an exceptional raw output. The placed circuit proof establishes the whole-register action and cleanup for this construction.

4.2 Register reuse

The Euclidean core has two 259-bit work words and 53 other positions. The packed remainders and coefficients occupy intervals of those work words during arithmetic. A separate 256-bit field and six auxiliary positions complete the 833-position allocation.

Quantum allocation	Positions
Two Euclidean work words	$2 \cdot 259 = 518$
Two 9-bit lengths, one 8-bit length, one 9-bit shift	35
Remainder-length extension	1
Two phase bits, iteration bit, and sign bit	4
Shared arithmetic and selector pool	13
Additional field register	256
Five-bit address and zero-factor flag	6
Total	833

Table 1: Arithmetic-stage allocation. Register contents and uses change between stages.

The point coordinates occupy the low 256 bits of the two work words at point-translation boundaries. During five-bit arithmetic, positions 827–831 hold the address and position 832 holds the zero-factor flag. After arithmetic restores its workspace, the circuit moves the address to positions 518–522. The released positions then hold selection and exceptional-case data. Correction and tag erasure finish before the address returns. The phase and length fields borrowed in this stage are restored.

Mutable classical positions 0–255 hold a reusable word of arithmetic measurement outcomes. Positions 256–511 and 512–767 hold the two Fourier results. Single-bit arithmetic erasures reuse position 767 before the last Fourier measurement writes it. The lifetime proofs preserve every earlier scalar-result bit. Mathematical branch histories in the semantics record all outcomes for probability accounting. The program’s declared classical storage holds the values needed for execution.

5 Branch semantics and measured distribution

For recorded Fourier outputs $(j, \ell) \in \{0, \dots, N-1\}^2$, define the subnormalized analytic branch state

$$|\psi_{j,\ell}\rangle = \frac{1}{N^2} \sum_{a,b=0}^{N-1} e^{2\pi i(aj+b\ell)/N} |\text{enc}([a+db]G), 0_{\text{work}}\rangle. \quad (6)$$

The state includes all 833 positions. The second component indicates the restored non-point quantum positions. The squared norm of (6) is the probability of (j, ℓ) in the ideal double-scalar Fourier experiment.

Let

$$M(m) = 2^{1+5m} - 2 + (103 - m) 44,426,048. \quad (7)$$

This is the number of internal measurements from the initial lookup and subsequent point translations. The complete branch theorem states that every program branch with scalar result (j, ℓ) has quantum state $2^{-M(m)/2}|\psi_{j,\ell}\rangle$, records those results in their declared classical positions, and preserves the immutable input. The structural branch-count theorem gives $2^{M(m)}$ histories for each scalar-result pair, counting the formal zero-weight branches as well.

Summing branch probabilities cancels the internal scale:

$$2^{M(m)} \left\| 2^{-M(m)/2} \psi_{j,\ell} \right\|^2 = \|\psi_{j,\ell}\|^2.$$

This identity connects the measurement-assisted program to its analytic output law. The proof uses symbolic sums and structural induction on the generated operations.

Define

$$\begin{aligned} F_N(j, t) &= \left| \frac{1}{N} \sum_{a=0}^{N-1} e^{2\pi i a(j/N - t/q)} \right|^2, \\ \mu_d(j, \ell) &= \frac{1}{q} \sum_{t=0}^{q-1} F_N(j, t) F_N(\ell, [dt]_q). \end{aligned} \tag{8}$$

Orthogonality of the characters of the order- q subgroup transforms the squared norm of (6) into (8). For each fixed t , the two Fourier-coordinate factors are nonnegative. Their finite sums have mass at most one, and the complete marginal has mass one. Lean proves pointwise equality between μ_d and the generated program's scalar-result distribution after summation over all auxiliary outcomes.

6 Bounded classical decoding and one-run recovery

The decoder searches a finite neighborhood of each measured Fourier coordinate. The proof bounds the probability outside those neighborhoods and establishes that any matching candidate gives the unique valid secret. It applies to every allowed d .

6.1 Fourier neighborhoods

For a subgroup label $0 \leq t < q$, define its nearest Fourier bin

$$J(t) = \left\lfloor \frac{2Nt + q}{2q} \right\rfloor.$$

Let $D_N(j, J)$ be the smaller distance between j and J around the cycle of N integers. The geometric-sum formula for F_N and the chord lower bound on sine imply, at cyclic distance $h \geq 1$ from $J(t)$,

$$F_N(j, t) \leq \frac{1}{4(h - \frac{1}{2})^2}.$$

For $h \geq 2$, this is at most $1/(4h(h-1))$. There are at most two integers at each cyclic distance. Therefore, for every integer radius $R \geq 1$,

$$\sum_{D_N(j, J(t)) > R} F_N(j, t) \leq \frac{1}{2} \sum_{h=R+1}^{\lfloor N/2 \rfloor} \left(\frac{1}{h-1} - \frac{1}{h} \right) \leq \frac{1}{2R}. \tag{9}$$

For a fixed nonzero label t , the product component in (8) places mass at most $1/R$ outside the rectangle of radius R around $(J(t), J([dt]_q))$. This follows by summing the two coordinate tails and

using the mass bound for the other factor. The zero-label component has total mass at most $1/q$. Summing over components gives

$$\Pr\{\text{output belongs to a nonzero-label rectangle}\} \geq 1 - \frac{1}{R} - \frac{1}{q}. \quad (10)$$

The rectangles may overlap. Their union still contains each retained component's contribution, which suffices for this bound.

6.2 Candidate generation and verification

For an integer Fourier coordinate $u \in \{0, \dots, N-1\}$, define

$$H(u) = \left\lfloor \frac{2qu + N}{2N} \right\rfloor \bmod q.$$

Given measured (j, ℓ) , form the lists

$$K = (H((j + i - R) \bmod N))_{i=0}^{2R}, \quad V = (H((\ell + i - R) \bmod N))_{i=0}^{2R}.$$

Compute $[k]Q$ for every entry $k \in K$ and $[v]G$ for every entry $v \in V$. Search their Cartesian product for

$$k \neq 0, \quad [k]Q = [v]G.$$

If a pair exists, return $[vk^{-1}]_q$. Otherwise return failure. The inverse is modulo q .

The order of G implies that a matching pair satisfies $kd \equiv v \pmod{q}$. Since q is prime and $k \neq 0$, every returned value equals d . The nearest-bin error satisfies $|qJ(t)/N - t| \leq q/(2N) < 1/2$, so $H(J(t)) = t$. An output in the radius- R rectangle for nonzero t therefore includes $k = t$ and $v = [dt]_q$, ensuring a match.

Taking $R = 128$ in (10) proves a success probability at least $1 - 1/128 - 1/q > 99/100$. Both candidate lists have 257 entries, giving exactly 514 scalar-multiplication calls, at most $257^2 = 66,049$ pair comparisons, and at most one inversion. These are bounds for the specified mathematical decoder. A classical implementation must also account for the group operations within scalar multiplication, field arithmetic, and stored candidate points.

7 Logical resource derivation

The resource theorems traverse the generated operations used by the execution proofs. Sequencing adds counts. Classical conditionals produce intervals between the possible path costs. For the CCZ metric, every conditional alternative has the same count, so the complete interval is a singleton. This proves equality of the pathwise count and, after normalization of the semantic probabilities, its expectation.

7.1 Arithmetic and complete execution

One *prepared traversal* consists of input preparation followed by the 1,620 Euclidean rounds. Division and multiplication each use a traversal in both directions. A point translation therefore contains four prepared traversals, together with terminal products, phase repair, and the remaining affine operations.

Five selected coordinate updates and the exceptional correction give the translation bound

$$P_5 = 2(83,941,623) + 3,703,905 + 1,536 + 5(5,795) + 392,960 = 172,010,622.$$

Component	Toffoli-class count
Prepared traversal, either direction	39,190,008
Complete in-place division	83,941,623
Complete in-place multiplication	83,941,623
Modular subtraction of a square	3,703,905
Modular negation	1,536
One selected coordinate update, including lookup and erasure	$\leq 5,795$
Five-bit exceptional correction and cleanup	392,960
Complete five-bit translation	$\leq 172,010,622$
Leading binary translation	171,630,174

Table 2: Syntax-derived component counts at the reported revision. The inequalities bound exact public-table-dependent counts.

Writing $P_1 = 171,630,174$, the complete scalar schedule costs at most $P_1 + (102 - m)P_5 + 2^k - 2$, which is (1). At $m = 5$, the 98 translations contain 392 prepared traversals. This accounts for the large arithmetic cost despite the five-bit scalar windows.

The exact count $C(Q, m)$ replaces each occurrence of P_5 by the count of its generated coordinate tables and correction circuit. That count can depend on the public point. The bound $B(m)$ holds uniformly over public-point values. Its finite difference is

$$B(m+1) - B(m) = -172,010,622 + 31 \cdot 2^{1+5m}.$$

The difference is negative through $m = 4$ and positive from $m = 5$. This proves that $m = 5$ minimizes the displayed upper bound. Lean also checks the minimum over all 26 supported values.

The structural measurement counts are 11,106,384 per prepared traversal and 44,426,048 per complete translation. Together with the initial lookup and 512 Fourier measurements, (7) gives

$$M(5) + 512 = 4,420,862,078$$

measurements per complete execution. Arithmetic erasure reuses its declared classical storage. Measurement latency and feed-forward therefore require attention in a physical implementation even though they have zero Toffoli weight.

7.2 Lookup width and classical storage

The initial lookup has the proved CNOT upper bound $514 \cdot 2^k - 2$. Its packed table contains 2^k point codes of 512 bits, or $64 \cdot 2^k$ bytes. These are separate from the Toffoli count and from the 833-position quantum allocation.

Prefix k	Complete Toffoli bound	Lookup CNOT bound	Packed table
11	17,372,694,420	1,052,670	128 KiB
16	17,200,747,286	33,685,502	4 MiB
21	17,030,768,280	1,077,936,126	128 MiB
26	16,923,769,370	34,493,956,094	4 GiB
31	18,832,133,532	1,103,806,595,070	128 GiB

Table 3: Verified lookup tradeoffs. Every row retains the allocation and recovery guarantee of Theorem 1.

The 21-bit choice increases the Toffoli bound by 106,998,910, about 0.63%, while reducing the packed initial table by a factor of 32. The 16-bit choice increases the bound by 276,977,916, about 1.64%, and uses a 4 MiB table. The Toffoli-minimizing prefix therefore selects one point in a broader resource tradeoff.

The proof keeps public tables and generated circuits symbolic. The table-size theorem counts packed point data. Materializing a compiler representation of the generated operations requires additional host storage. Public preprocessing time, all later small tables, total CNOTs, and a complete circuit-depth bound need separate accounting. The construction assumes classical compilation of table data into controlled operations.

8 Formal verification and its scope

VQ defines reversible gate action, hybrid quantum execution, and resource functions in Lean. A hybrid branch contains a subnormalized quantum state, mutable classical storage, immutable input, and an outcome history. The execution semantics include projection, reset, and classical feed-forward. Exact dyadic cyclotomic amplitudes map to complex amplitudes, supplying the conventional Born probabilities. A phase level $\lambda \geq 257$ supplies the roots of unity required by the 256-bit Fourier operations.

The proof proceeds from placed bit operations to arithmetic invariants, complete point translation, the two scalar loops, the terminal branch state, and the observed probability law. Each composition establishes the workspace and input conditions required by its next component. The recovery theorem then applies the analytic argument to the distribution of that same program. Resource equalities refer to its operation list.

The complete-result build checked the execution, recovery, and resource modules identified in Appendix C, together with their dependencies. The endpoint axiom guards permit only propositional extensionality, classical choice, and quotient soundness, named `propext`, `Classical.choice`, and `Quot.sound` in Lean. Exact finite checks and analytic inequalities support the numerical bounds. The report uses the recorded successful build at the pinned revision and a source-statement and arithmetic review. It introduces no changed algorithm or new Lean proof.

The kernel checks proof terms against the formal definitions. Interpreting those definitions as the intended quantum algorithm also depends on the gate semantics, encoding, initialization, and compiler model. Appendix C identifies the statements that connect these objects, allowing their hypotheses and conclusions to be inspected separately.

An earlier binary-scalar program at revision `9d865f6` passed statement comparison and independent Lean and unchanged NanoDa checks [9]. That program had 588,551,462,912 Toffolis per execution and a 26-run selected-outcome recovery guarantee. Those independent-checker results apply to that earlier program. The present five-bit program’s reported verification is the Lean build and endpoint axiom audit described above.

9 Comparison with leading constructions

Published resource estimates differ in curve, success probability, classical recovery work, arithmetic error, and amortization. Table 4 preserves those distinctions. The comparison uses the versions current on 19 September 2026. An execution count, a sample average, and a bound on every path support different conclusions about resource use.

Table 4: Reported logical resources. Counts are Toffoli-class operations except the explicitly marked T-gate rows.

Construction	Qubits	Count or bound	Scope
Roetteler et al. 2017 [10]	2,330	$1.26 \cdot 10^{11}$	P-256 resource estimate from arithmetic circuits.
Häner et al. 2020 [11]	2,124 / 2,619	$1.72 \cdot 2^{32} / 1.08 \cdot 2^{31}$ T gates	P-256 low-width / low-T variants.
Litinski 2023 [12]	$3,000h$	$(44 + 65/h) \cdot 10^6$ per key	h parallel instances, shared inversion and state reuse.
Litinski 2024 [13]	About $3,000h$	$(30 + 73/h) \cdot 10^6$ per key	Revised multiplication in the amortized construction.
Chevignard et al. 2026 [14]	1,193	$2^{38.98}$ per run	P-256, average 22 runs.
Babbush et al. v2 [15]	$< 1,200 / < 1,450$	$< 90 \cdot 10^6 / < 70 \cdot 10^6$	secp256k1 estimates using tested private circuits.
Schrottenloher v1 [16]	1,208 / 1,462	About $73 \cdot 10^6 / 58 \cdot 10^6$	secp256k1 approximate arithmetic.
Luo et al. v3 [19]	835 claimed	About $1.977 \cdot 10^9$	secp256k1 estimate from point-addition cost and window formula.
Häner et al. September 2026 [18]	1,457 reported	About $39\text{--}40 \cdot 10^6$	secp256k1 approximate arithmetic and physical compilation study.
VQ recorded-history program [22]	1,488	$\leq 257,008,864$	Complete exact secp256k1 program, one-run recovery $> 99/100$.
This report, 26-bit lookup	833	$\leq 16,923,769,370$	Complete exact secp256k1 program, one-run recovery $> 99/100$.

9.1 Circuit simulation and published resource formulas

Roetteler–Naehrig–Svore–Lauter implemented reversible arithmetic in LIQUi| $\rangle$ and simulated controlled point addition at cryptographic parameter sizes. This is simulation and resource-counting evidence for the arithmetic construction [10, Sections 4–5]. Häner–Jaques–Naehrig–Roetteler–Soeken provide $Q\#$ arithmetic and point-addition implementations with unit tests and automated resource estimation. Their optimization uses windowed arithmetic and changed uncomputation schedules. Their published comparison counts T gates, with decomposition and measurement choices included in that metric [11, Section 6 and Table 1]. Converting those totals into a Toffoli column requires the specified gate decomposition and its full cost model.

Litinski’s estimates combine circuit cost formulas with a fault-tolerant architecture model. The per-key formulas in Table 4 charge shared inversion across h instances and reuse of prepared state. The 2024 paper reduces the arithmetic cost with a schoolbook multiplier based on conditional addition/subtraction. These papers provide circuit derivations and resource analysis. Their advertised

50-million and 30-million regimes involve amortization across keys, whereas the present theorem bounds one initialized execution for one key [12, 13].

Chevignard–Fouque–Schrottenloher compress the group output to a Legendre-symbol computation using a residue-number representation and a reversible storage schedule. Their evidence includes mathematical arguments with stated heuristics, implementation of the storage strategy, and public Qarton code. Table 8 reports 1,193 qubits and $2^{38.98}$ Toffolis per P-256 execution, with 22 executions on average. The 1,098-qubit figure in the same table is for P-224 [14, Sections 3–6 and Table 8]. Keeping the curve and repetition count attached to those figures avoids comparing different parameter sizes as one result.

9.2 Cryptographic attestation of private circuit tests

Babbush et al. publish an SP1/Groth16 proof that a committed private point-addition circuit passed 9,024 hash-derived tests with the specified resource checks. Their simulator tracks basis values and measurement-induced phases. The point-circuit limits are 1,175 qubits and 2.7 million counted non-Clifford gates, or 1,425 qubits and 2.1 million gates. Appendix A specifies that the gate count is the average executed count over those tests. The complete estimates add lookups and classical recovery [15, Appendix A].

The attested proposition concerns that test procedure on the committed circuits. Extending sampled success to untested inputs uses its sampling argument and cryptographic assumptions. Relating sample-average cost to expected quantum cost requires a distributional argument. Version 2 corrects an earlier soundness defect. We did not run the published verifier [15].

9.3 Open approximate circuits and statistical guarantees

Schrottenloher publishes Qarton circuits for in-place arithmetic, compressed Euclidean history, and windowed point addition. The paper’s grouped count includes CCX, CCZ, and forward temporary AND operations. Its complete estimates use 28 additions with 16-bit windows. The test implementation replaces exact arithmetic subcircuits by their classical functions during simulation. Consequently, its random-input tests cover a composed arithmetic model with those substitutions. The paper reports a point-test error figure and a full-algorithm estimate [16, Sections 1–2, Tables 1–2].

The relation between a component’s test-input law and the scalar algorithm’s coherent call distribution is a separate probability question. Measurement cleanup also requires the phase action on the same inputs. VQ’s version-specific audit proves component results under their stated premises and records termination, shrinking-register, and resource-accounting counterexamples and discrepancies [17]. Those findings motivate explicit source versions and complete caller statements in this comparison.

Häner et al.’s September 2026 construction adds arithmetic optimizations and randomized point offsets. It reports a logical success lower bound of 0.553, averaged over fresh offsets, with confidence at least $1 - 2^{-128}$ from its arithmetic sampling and analysis. Its 0.861 estimate uses a heuristic ideal-recovery assumption. The 39-million core estimate substitutes the largest point cost observed in 100 runs into $28(1,196,000 + 3 \cdot 2^{16})$. Section XI describes exhaustive small-component checks, sampled large-component checks, phase-aware checks, and tests of the compiled measurement schedules [18, Sections II–IV, XI, and Appendix B].

Conditional on the 0.553 success bound, six independent executions with fresh offsets give $1 - (1 - 0.553)^6 > 0.992$, at about 234–240 million reported Toffolis, inheriting the source’s arithmetic

and cost qualifications. Its 28-addition schedule also transfers 48 scalar bits to classical recovery, whose work differs from the bounded decoder used here.

9.4 Packed inversion and formal verification

Luo et al. v3 report a shared-register Euclidean inverter, a 70.29-million-Toffoli point adder, and an 835-qubit allocation claim. Substitution into their full-ECDLP formula gives

$$28(70,290,000 + 5 \cdot 2^{16}) = 1,977,295,040.$$

The input point cost is rounded, so the result is an estimate. Their evidence includes mathematical schedule arguments, Qiskit subblock generation and counting, and representative tests of arithmetic and assembly [19, Sections 6.3–6.4].

VQ’s earlier audit of v2 identifies repairs to coefficient endpoints, terminal behavior, and caller composition, with versioned source and Lean evidence [20]. The v3 comparison records the revised coefficient endpoint and window formulas, while retaining explicit terminal extraction, zero-endpoint, complete point-addition, and lookup-allocation obligations [21]. The 833-qubit theorem applies to the reconstructed VQ program and its specified operation list. A source-correspondence proof would be needed to transfer that theorem to a Luo publication or companion generator.

The separate VQ recorded-history construction uses 1,488 quantum positions and at most 257,008,864 Toffolis, with complete execution, cleanup, and one-run recovery above 99/100 checked in Lean [22]. It illustrates the cost of the tighter 833-qubit allocation within the present implementations. The packed program repeatedly selects intervals through coherently stored length metadata. The wider construction retains a Euclidean decision history and uses a different arithmetic schedule. The current results establish these two resource points. An optimal time-space tradeoff remains open.

10 Limits of the resource theorem

The theorem assumes ideal logical gates and exact Fourier phases. Replacing rotations by a finite fault-tolerant gate set requires a synthesis-error bound and its cost. Arithmetic has zero algorithmic error on the specified valid inputs, but physical noise requires a separate error-correction and decoding model. The report therefore supplies no physical-qubit count or elapsed-time estimate.

The reported 833 is an exact declared allocation and a proved peak-live upper bound. The proof establishes sufficient storage for this program. Allocation minimality, complete CNOT and depth counts, and the resource cost of a materialized compiler output remain open. The decoder bounds count group-level operations, while a full classical cost model must include their implementation. These qualifications determine which entries of Table 4 permit direct numerical comparison.

A secp256k1 parameters

The field prime, subgroup order, and generator coordinates below use hexadecimal notation. Spaces separate 32-bit groups and carry no numerical meaning. They are the secp256k1 parameters of SEC 2, version 2.0, Section 2.4.1 [4].

p	FFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFE FFFFC2F
q	FFFFFFFF FFFFFFFF FFFFFFFF FFFFFFFE BAAEDCE6 AF48A03B BFD25E8C D0364141
G_x	79BE667E F9DCBBAC 55A06295 CE870B07 029BFCDB 2DCE28D9 59F2815B 16F81798
G_y	483ADA77 26A3C465 5DA4FBFC 0E1108A8 FD17B448 A6855419 9C47D08F FB10D4B8

The cofactor is one. The VQ proofs establish primality and the generator-order fact used in recovery. The encoded-key hypothesis requires either the identity code or a reduced on-curve coordinate pair. The additional $0 < d < q$ hypothesis in the complete theorem excludes the identity as the public key.

B Euclidean weight certificate

The 1,620-round bound can be expressed through a finite linear certificate. Put $\rho = 2 + \sqrt{3}$ and $\eta = \rho^{1/3}$, and define the four row vectors

$$u_0 = (\rho - 1, 1), \quad u_1 = \eta^{-1}(\rho, \rho - 1), \quad u_2 = \eta^{-1}(\rho + 1, 1), \quad u_3 = \eta^{-2}(\rho + 2, \rho + 1).$$

For $a \geq b \geq 0$, let $A(a, b) = \min_{0 \leq i < 4} u_i \cdot (a, b)$. The checked Euclidean-step inequalities imply

$$\eta^{W(a,b)} A(\gcd(a, b), 0) \leq A(a, b).$$

They compare the four rows after the quotient matrix $\begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix}$ with the original rows on this cone, using $c \geq 2^{\lceil \log_2 c \rceil}$. The initial and terminal bounds are

$$A(1, 0) \geq 2, \quad 2A(p, x_0) \leq (2\rho - 1)p \quad \text{when } 2x_0 \leq p.$$

The exact numerical certificate is

$$(2\rho - 1)2^{256} < 4\eta^{406}.$$

Since $\gcd(p, x_0) = 1$ and $p < 2^{256}$, these inequalities give $\eta^{W(p, x_0)} < \eta^{406}$ and hence $W(p, x_0) \leq 405$. The arithmetic invariant connects this weight bound to the four-phase schedule and its terminal padding. The source contains the row comparisons, rational bounds for the algebraic constants, and the transition proof [1].

C Checked statements and reproduction

All source links in Table 5 refer to revision

c46c28927244d3cb1ee664677a0f61145ca45d54.

Module names identify files under the matching namespace directory. The linked declarations provide their full hypotheses. The complete execution and resource targets import the supporting arithmetic and probability statements.

Table 5: Principal formal statements.

Claim	Module and declaration
Program and valid indices	MeasuredFiveBitLookupProgram: program, program_wellFormed.
Complete execution and recovery	MeasuredFiveBitLookupRecovery: execution, bounded_recovery_probability.
Observed probability law	MeasuredFiveBitLookupMarginal: distribution_eq_analytic, runOps_pair_count, internal_scale_cancel.
Path and expected costs	MeasuredFiveBitLookupResources: program_toffoli, path_toffoli, expected_toffoli, expected_toffoli_upper, upperBound_five_minimum.
Classical decoding work	WindowRecovery.bounded_decoder_resources.
Analytic probability bound	BoundedProbability: boundedContribution_lower_bound.
Decoder correctness	BoundedDecoder: boundedDecode_recovers_of_near_peaks, boundedDecode_resources_128.
Complete selected translation	MeasuredAffineFiveBitTranslation.ops_correct_groupAdd.
Translation cost	MeasuredAffineFiveBitResources: translationCount_le, program_toffoli.
Field arithmetic costs	MeasuredAffineArithmeticResources: division_toffoli, multiplication_toffoli.
Fixed schedule	LuoScheduleBound: preprocessed_euclidWeight_le_405.
Prepared measured schedule	MeasuredOwnershipPrefixPrepared: prepared_forward, prepared_backward, in namespace LuoSchedule.MeasuredOwnershipPrefix.
Totalized field action	MeasuredAffineTotal: division, multiplication.
Initial table and its size	FiveBitLookupTable: table, table_length, complete_terms.
Lookup storage and CNOTs	FiveBitLookupResources: publicTableBits_eq, lookup_cnot_upper. The measured program uses the same initial lookup.

The artifact pins Lean 4.34.0-rc2 and the following dependencies:

```
Mathlib      85e3a25e006c35636f0e53b0e9296caca2685bc0
hex-matrix   d45bf749d6283726b8df43b5be9e6dde350666d6
```

The repository includes the dependency manifest, source, and resource-limited execution procedure. In a checkout of the pinned revision, the relevant Lake command is:

```
lake --no-ansi --rehash --reconfigure build \
  VQMathlib.Euclid.MeasuredOwnershipPrefixPreparedResources \
  VQMathlib.ECDLP.PackedAffine.MeasuredFiveBitLookupRecovery \
  VQMathlib.ECDLP.PackedAffine.MeasuredFiveBitLookupResources
```

The recorded run used the repository’s persistent Lean runner on `dev`, with 20 GiB memory-high and 24 GiB memory-maximum limits, and finished with status zero. Its identifier is `vq-833opt-high-endpoint-complete-20260910-1`. The development journal records the source checksum comparison and endpoint axiom checks [1]. A fresh reproduction requires the pinned dependencies and enough memory for the aggregate. The source definitions generate the large operation lists symbolically.

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