

Toffoli Reductions and Single-Run Recovery for an 833-Qubit secp256k1 ECDLP Program

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Abstract

We verify an 833-logical-qubit secp256k1 discrete-logarithm program that recovers every valid nonzero private scalar with probability greater than 99 percent in one execution and uses at most 36,591,123,006 Toffoli-class operations on every path and in expectation. Starting from the previously reported bound of 588,551,462,912 operations per execution, we derive reductions from shared Euclidean endpoint comparisons, scalar windows, output-side recognition of exceptional point additions, and an initial public-point lookup. An analytic Fourier-tail bound supports a bounded classical decoder and replaces the earlier 26-execution recovery guarantee. The smallest proved Toffoli bound in the resulting lookup family uses a 26-bit prefix, a 4 GiB packed public table, and at most 34,493,956,094 lookup CNOTs. Smaller prefixes reduce these classical-storage and CNOT costs. Lean proofs connect the generated operations, restored workspace, exact branch amplitudes, recovery probability, and logical resources. Structural count lemmas and symbolic scalar and measurement-history decompositions reduce the finite computations required in those proofs.

1 Complete-program bounds

The verified program recovers a secp256k1 private scalar with probability greater than 99/100 in one execution, using an allocation of 833 logical qubits and at most 36,591,123,006 Toffoli-class operations. It computes the double-scalar map used in elliptic-curve Fourier sampling [2, 3]. Let $E/\mathbb{F}_p$ be secp256k1, $y^2 = x^3 + 7$, where $p = 2^{256} - 2^{32} - 977$. Let G be its standard generator of prime order q , and let $Q = [d]G$ be a canonically encoded public point with $0 < d < q$. Set $N = 2^{256}$. The coherent scalar expansion contains every $(a, b) \in \{0, \dots, N - 1\}^2$ and has point value $[a]G + [b]Q = [a + db]G$. Semiclassical Fourier transforms measure the scalar registers as their qubits become available for reuse.

Theorem 1 (Five-bit windows with initial lookup). *For every such Q and every integer $0 \leq m \leq 25$, the generated program with initial prefix width $k = 1 + 5m$ has quantum allocation 833 and mutable classical allocation 768 bits. Starting with zero quantum and mutable classical storage, and using exact semiclassical Fourier rotations, its scalar-output distribution is the Fourier distribution in (5). Every branch restores the arithmetic and address workspace. The decoder in Section 3.2 returns d with probability greater than 99/100. Let T denote the execution’s Toffoli-class count. There is a public-point-dependent integer $C(Q, m)$ such that every execution path has exactly $C(Q, m)$ Toffoli-class operations, and*

$$\mathbb{E}[T] = C(Q, m) \leq B_5(m) := 372,317,456 + (102 - m)372,697,904 + 2^{1+5m} - 2. \quad (1)$$

The minimum of $B_5(m)$ on this integer interval occurs at $m = 5$.

The expectation equality extends to every basis input within the allocated quantum width, with zero initial mutable storage. Recovery uses the initialized state in Theorem 1. The 768 classical bits cover the quantum program’s measurement and feed-forward storage. Public tables and the decoder require separate classical memory.

One Toffoli-class operation means one compiled CCZ. The compiler represents a reversible Toffoli by conjugating that CCZ with target Hadamards. The metric assigns zero cost to measurements, resets, classically selected Paulis, and Fourier phase rotations. A fixed 833-qubit allocation bounds peak live logical qubits by 833. It includes point coordinates, packed field-arithmetic state, scalar addresses, exceptional-case data, and temporary storage.

The binary baseline in Table 1 is the endpoint of the preceding report [1]. Its Euclidean coefficient bounds and round-indexed search intervals remain in use. Exact binary counts are independent of the public point. Windowed counts have exact table-dependent expressions, for which the table gives uniform upper bounds.

Construction	Complete-program Toffolis	Count
Previous report	588,551,462,912	exact
Direct controlled endpoint equality	376,880,472,064	exact
Shared endpoint-writer prefixes	253,849,772,032	exact
Shared remainder endpoint blocks	190,626,537,472	exact
Two-bit scalar windows	95,326,030,336	upper bound
Three-bit scalar windows	64,061,309,332	upper bound
Five-bit scalar windows	38,759,821,120	upper bound
Five-bit windows with 26-bit initial lookup	36,591,123,006	upper bound

Table 1: Checked per-execution Toffoli counts at quantum allocation 833. The recovery improvements in Section 3 apply to the common Fourier distribution.

VQ is a Lean 4 framework for exact functional verification and formally certified logical-resource analysis of quantum algorithms for cryptography and cryptanalysis [4]. Its semantics retain complex amplitudes and measurement branches, including reset and classical feed-forward. In this development, proofs connect reversible arithmetic to complete group operations and connect the emitted hybrid program to its Fourier distribution. Resource theorems count operations and bound allocation for the same program. Mathlib supplies the finite-sum, complex-analysis, modular-arithmetic, and elliptic-curve results.

2 Shared Euclidean endpoint comparisons

The packed field operations retain the fixed 1,620-round Euclidean schedule of the previous report. Input preparation normalizes a nonzero field operand, computes its length, and initializes the packed modulus and control fields. The *prepared forward schedule* consists of this preparation followed by the Euclidean rounds, including local scratch restoration within each component. Terminal extraction, reversal of the prepared schedule, measured cleanup, and phase correction are counted separately in the complete field operation.

The reversible endpoint searches compare a candidate position with a stored endpoint and use the comparison to select a bit update. Adjacent positions share endpoint-address prefixes. Computing those prefixes once per interval reduces the comparison cost while preserving the order of the arithmetic updates.

2.1 Direct equality and endpoint writers

A nine-bit equality computed into a temporary, used under an additional control, and erased costs $15 + 1 + 15 = 31$ Toffolis. Combining the equality and outer control in a ten-control conjunction costs 17. The replacement has the same controlled action and restores its comparison scratch. It reduces the prepared forward-schedule count from 267,127,424 to 163,772,448.

An endpoint writer records a selected bit position in a length field. It shares comparison work across candidate positions. Suppose a temporary contains $c \wedge \neg b$, where c is the predicate for an address prefix and b is the next address bit. A CNOT controlled by c changes that temporary to $c \wedge b$:

$$(c \wedge \neg b) \oplus c = c \wedge b.$$

This identity replaces separate erasure and preparation between sibling prefixes, using the prefix-reuse principle of unary iteration [5]. A recursive traversal computes a prefix, visits its negative and positive children in order, and restores the prefix temporary. For all w -bit addresses its selector cost satisfies

$$A(0) = 0, \quad A(w + 1) = 2A(w) + 2, \quad A(w) = 2^{w+1} - 2.$$

Thus a nine-bit traversal costs 1,022 selector Toffolis, compared with $512 \cdot 17 = 8,704$ for independent controlled equalities. Leaf actions contribute separately.

The generated endpoint intervals prune this traversal to the required leaves. The proof retains their ordering and proves restoration of every prefix temporary. A full length writer costs 6,258 Toffolis, compared with 21,758 before prefix sharing. Composing the pruned writers over the round-indexed intervals gives prepared-schedule count 103,698,864.

2.2 Chunked remainder updates

The remainder pass compares each of 259 bit positions with two endpoints. The checked construction splits each nine-bit endpoint address into six high bits and three low bits. Within a block of eight positions, it retains both high-prefix predicates and recomputes the low comparisons. Two high-prefix flags and seven comparison-scratch bits fit the existing temporary allocation.

A high-prefix computation costs 11 Toffolis, and a low comparison costs 5. Computing and erasing both high flags costs 44 per block and direction. The pass visits 33 blocks, with a final block of three positions. Both traversal directions together use four low comparisons per position and seven carry-update Toffolis per position. The count is

$$4 \cdot 5 \cdot 259 + 2 \cdot 33 \cdot 44 + 7 \cdot 259 = 9,897. \quad (2)$$

The earlier direct-comparison pass costs 19,425. The changed pass has checked forms that update or preserve a separate sign bit, together with reverse and inactive-control proofs. The sign-updating form inserts a carry-to-sign CNOT between the scans. The sign-preserving form leaves that bit unchanged. Both forms have the Toffoli count in (2), permit arbitrary incoming carry values, and restore the endpoint selectors.

The fixed portion of a Euclidean round falls from 48,397 to 29,341 Toffolis. Together with the shared endpoint writers, this gives prepared forward-schedule count $S = 72,828,144$. The retained field division and multiplication each cost

$$D = 2S + 30,323,778 = 175,980,066.$$

The complete controlled point translation has count

$$P_1 = 4S + 81,004,880 = 372,317,456. \quad (3)$$

The binary scalar program uses 512 such translations. The same formula relates each prepared-schedule count above to its complete-program count in Table 1. The proofs compose forward arithmetic, measured cleanup, phase correction, and exceptional-case correction, preserving the translation's coherent branch action.

3 Analytic recovery

All scalar-window constructions have the same measured scalar-pair law. For $0 \leq j < N$ and $0 \leq t < q$, define

$$F_N(j, t) = \left| \frac{1}{N} \sum_{s=0}^{N-1} \exp\left(2\pi i s \left(\frac{j}{N} - \frac{t}{q}\right)\right) \right|^2, \quad (4)$$

$$\mu_d(j, \ell) = \frac{1}{q} \sum_{t=0}^{q-1} F_N(j, t) F_N(\ell, [dt]_q). \quad (5)$$

Here $[x]_q$ is the least nonnegative residue modulo q . The group-character calculation for $[a + db]G$ gives (5). The two Fourier coordinates are independent conditional on a component t . The program proof identifies its marginal with this normalized finite mixture.

3.1 Averaged nearest-peak bounds

Let

$$J(t) = \left\lfloor \frac{2Nt + q}{2q} \right\rfloor, \quad a_t = F_N(J(t), t), \quad \bar{a} = \frac{1}{q} \sum_{t=0}^{q-1} a_t.$$

Each nonzero t gives a decodable selected pair $(J(t), J([dt]_q))$. These pairs are distinct because $N > q$. Their total probability p_{sel} is at least $q^{-1} \sum_{t=1}^{q-1} a_t a_{[dt]_q}$. Since $0 \leq a_t \leq 1$, $a_0 = 1$, and multiplication by d permutes the nonzero residues, the inequality $ab \geq a + b - 1$ gives

$$p_{\text{sel}} \geq 2\bar{a} - 1 - \frac{1}{q}. \quad (6)$$

The centered rounding offsets $x_t = J(t) - Nt/q$ permute the values r/q for $-(q-1)/2 \leq r \leq (q-1)/2$. This follows from oddness of q and coprimality of N and q . Consequently their moments are exact:

$$\mathbb{E}[x_t^2] = \frac{q^2 - 1}{12q^2}, \quad \mathbb{E}[x_t^4] = \frac{(q^2 - 1)(3q^2 - 7)}{240q^4}. \quad (7)$$

The finite geometric sum in (4) implies $a_t \geq \text{sinc}(\pi x_t)^2$, with $\text{sinc } z = \sin z/z$ and $\text{sinc } 0 = 1$. A rational polynomial envelope on $|x| \leq 1/2$ is

$$\text{sinc}(\pi x)^2 \geq (1 - 5x^2/3)^2.$$

Combining it with (7) proves $\bar{a} \geq 109/144$ and $p_{\text{sel}} > 1/2$ for the secp256k1 order. Seven independent executions therefore suffice for recovery above 99/100.

A sharper proved envelope is

$$\text{sinc}(\pi x)^2 \geq 1 - \frac{484}{147}x^2 + \frac{32}{9}x^4 \quad (|x| \leq 1/2).$$

The proved Taylor inequality

$$\text{sinc}(\pi x)^2 \geq 1 - \frac{\pi^2 x^2}{3} + \frac{2\pi^4 x^4}{45} - \frac{\pi^6 x^6}{315}$$

and $x^2 \leq 1/4$ give a quartic coefficient $2\pi^4/45 - \pi^6/1260$. The rational bounds $157/50 < \pi < 22/7$ place it above $32/9$ and place $\pi^2/3$ below $484/147$. The resulting mean gives $\bar{a} \geq 566/735$, and (6) yields $p_{\text{sel}} > 27/50$. Six executions suffice because $(23/50)^6 < 1/100$. These arguments strengthen the analysis of the selected outcomes without changing the quantum computation. The previous report's pointwise bound required 26 executions.

3.2 Cyclic neighborhoods and bounded decoding

Accepting nearby Fourier labels gives a one-execution bound. Write $D_N(j, J(t))$ for cyclic distance modulo N and take an integer $R \geq 1$. At distance $h \geq R + 1$, the geometric-sum formula and the chord bound on sine imply

$$F_N(j, t) \leq \frac{1}{4h(h-1)}.$$

There are at most two indices at each distance. Telescoping the finite sum proves

$$\sum_{D_N(j, J(t)) > R} F_N(j, t) \leq \frac{1}{2} \sum_{h=R+1}^{\lfloor N/2 \rfloor} \left(\frac{1}{h-1} - \frac{1}{h} \right) \leq \frac{1}{2R}. \quad (8)$$

Each coordinate distribution has total mass at most one, so its product component places at most $1/R$ outside the radius- R rectangle about $(J(t), J([dt]_q))$. Summing componentwise in (5) and discarding the component $t = 0$, of mass at most $1/q$, gives

$$\mathbb{P}\{\text{outcome lies in a nonzero component's rectangle}\} \geq 1 - \frac{1}{R} - \frac{1}{q}. \quad (9)$$

Overlapping rectangles preserve this lower bound: each retained component contribution lies in the union, and all summands are nonnegative.

For measured (j, ℓ) , the decoder enumerates the $2R + 1$ cyclic neighbors of each coordinate and rounds their products with q/N to residues modulo q . Let K and V be the resulting lists. It precomputes $[u]Q$ for $u \in K$ and $[v]G$ for $v \in V$. For a pair satisfying

$$u \neq 0 \pmod{q}, \quad [u]Q = [v]G,$$

it returns $vu^{-1} \pmod{q}$. The order of G and $Q = [d]G$ force $v \equiv ud \pmod{q}$, so every returned value equals d . Since $|qJ(t)/N - t| \leq q/(2N) < 1/2$, rounding $qJ(t)/N$ recovers t . An outcome in the rectangle of a nonzero component includes $u = t$ and $v = [dt]_q$ among its candidates, and therefore yields such a pair.

With $R = 128$, (9) exceeds 99/100. The decoder uses 514 scalar-multiplication calls, at most 66,049 candidate comparisons, and at most one modular inversion. These are proved bounds on the mathematical decoder. Field-operation costs inside the scalar multiplications require a separate classical implementation analysis. The probability argument uses exact finite sums and analytic inequalities throughout.

4 Scalar windows and coherent offsets

A width- w digit $r \in \{0, \dots, 2^w - 1\}$ selects the public point $[r + 1]P$. The nonzero digit table keeps every selected addend finite and nonzero. A public correction to the initial point cancels the extra P from each digit. The circuit selects the addend's coordinates and applies one shared field-arithmetic computation. Its arithmetic cost therefore grows slowly with the table width, while the number of point translations falls. Public-table scalar windows and replacement of initial additions by lookup follow the high-level reductions described by Schrottenloher [6, Section 2]. Their placement here must preserve the complete point-addition semantics within 833 qubits.

For a scalar with n complete width- w digits, define

$$s_{w,n} = \sum_{j=0}^{n-1} 2^{wj} = \frac{2^{wn} - 1}{2^w - 1}.$$

The digit bases are $[2^{wj}]G$ in the first scalar loop and $[2^{wj}]Q$ in the second. Initializing the point to $-[s_{w,n}](G + Q)$ cancels all digit shifts. For $w = 2$, $n = 128$ covers each scalar. For $w = 3$ or 5 , $256 = 1 + wn$, so each scalar begins with a binary digit followed by 85 or 51 complete digits. The leading binary step uses the ordinary controlled translation and adds no digit shift.

The two-bit construction replaces four exceptional-case flags with a three-bit code for the ordinary case and four exceptional cases. This frees an address position. The three-bit construction also derives the zero-factor predicate from that case code and the selected public addend. For addend (A, B) and input (x, y) , division uses factor $x - A$. Multiplication uses factor $x - A - \lambda^2 + 3A$, where $\lambda = (y - B)/s(x - A)$, $s(0) = 1$, and $s(z) = z$ for $z \neq 0$. Each predicate tests whether its factor is zero. For an ordinary input both factors are nonzero. For an exceptional input the factors are public functions of the case and addend. The Boolean lookup XORs the predicate into the least significant bit of the factor register, qubit 0. It maps a zero factor to one and preserves a nonzero factor. Reversing the lookup after arithmetic restores the original factor. Both queries restore their scratch and preserve the digit address.

For every address in superposition, the selected translation returns the correct group sum with restored workspace and a common internal branch amplitude. The scalar proof then separates the source scalar, the accumulated group point, and the Fourier phase. Appending a digit at weight 2^e adds $r2^e$ to the source label. Range-block identities enumerate each scalar in $\{0, \dots, N - 1\}$ exactly once. The semiclassical measurement rule extends the recorded result and its phase, and clears the digit address. Public-offset cancellation leaves point label $[a + db]G$ after both loops. Summing internal measurement records yields (5).

The checked selected-translation bounds are $P_2 = 372,367,306$ and $P_3 = 372,451,026$. Thus the two-bit program costs at most $256P_2$, and the three-bit program at most $2P_1 + 170P_3$. Extending to five address bits requires changing when exceptional inputs are identified.

5 Output-side exceptional recognition

Exceptional point-addition cases can be identified from the raw arithmetic output. This permits the case bits to serve as digit-address bits while the field arithmetic runs. The proof uses an inverse of the totalized affine arithmetic on all coordinate pairs.

Fix a finite nonzero public addend $P = (A, B)$, with s as defined in Section 4. For arbitrary

$(x, y) \in \mathbb{F}_p^2$, the raw map computes

$$\begin{aligned} \delta &= x - A, & \lambda &= (y - B)/s(\delta), & u &= \delta - \lambda^2 + 3A, \\ X &= A - u, & Y &= \lambda s(u) - B. \end{aligned} \tag{10}$$

An inverse is

$$\begin{aligned} u &= A - X, & \lambda &= (Y + B)/s(u), & \delta &= u + \lambda^2 - 3A, \\ x &= \delta + A, & y &= \lambda s(\delta) + B. \end{aligned} \tag{11}$$

Both divisions have nonzero denominators by definition of s . Substitution shows that (11) recovers (x, y) from (10). The raw map is therefore injective on $\mathbb{F}_p^2$, including off-curve coordinate pairs.

The complete group operation has exceptional inputs $\mathcal{O}$, P , $-P$, and $-2P$, encoding $\mathcal{O}$ by $(0, 0)$. These input codes are distinct because P has prime order $q > 3$ and $(0, 0)$ lies off the curve. Their raw outputs are public and distinct. Injectivity proves that equality with one of those outputs identifies exactly the corresponding exceptional input. After the raw arithmetic, the circuit compares against these outputs, records the case, and corrects to the group sum. The distinct corrected outputs then permit erasure of the case code. Ordinary valid inputs follow the affine addition formula and match none of the exceptional raw outputs. Equation (11) supplies the identification proof. The generated correction uses comparisons and public-coordinate updates.

The five-bit address occupies qubits 827–831 during arithmetic, and the zero-factor flag occupies qubit 832, using zero-based indices. After the arithmetic restores its temporary coordinate workspace, the address moves to positions 518–522. The released auxiliary positions hold row-selection, case, and equality data during correction. The circuit then restores the address and clears those temporary values. The placement proofs establish the required disjointness at each stage.

A selected 256-bit coordinate lookup costs at most 640 Toffolis. Loading, adding or subtracting, and erasing the selected coordinate costs at most 22,842. Retained division and multiplication each cost D from Section 2. The resulting five-bit selected translation has upper bound

$$\begin{aligned} P_5 &= 2D + 20,229,066 + 1,536 + 5 \cdot 22,842 + 392,960 \\ &= 372,697,904. \end{aligned} \tag{12}$$

The remaining terms respectively count the square-subtract block, negation, selected-coordinate operations, and exceptional correction. The proof covers coherent selection, the internal measurement phases, all point-addition cases, and workspace restoration. Each 256-bit scalar uses one leading binary translation and 51 five-bit translations, giving $2P_1 + 102P_5 = 38,759,821,120$ as the complete-program upper bound.

6 Initial public-point lookup

The first scalar loop prepares its initial point values directly from a public table. For a prefix of k scalar bits, the program selects a precomputed 512-bit point code, applies the semiclassical Fourier transform to the prefix address, and continues with the remaining scalar digits. Public-table initialization replaces the point translations for that prefix. The second scalar loop still acts on the point superposition produced by the first.

For the five-bit construction, set $k = 1 + 5m$ and

$$s = \sum_{j=0}^{50} 32^j, \quad D_m = \sum_{j=51-m}^{50} 32^j, \quad P_{\text{init}} = -[s](G + Q),$$

where $D_0 = 0$. The table entry at $0 \leq v < 2^k$ is

$$P_{\text{init}} + [v2^{256-k} + D_m]G. \quad (13)$$

The D_m term accounts for the nonzero digit-table shifts already consumed by the prefix. Completing the remaining digits gives the same scalar labels and group values as the full first scalar loop. The table permits the identity and uses the program’s canonical point encoding for every entry.

6.1 Lookup cost and branch amplitudes

The lookup traverses the binary address prefixes and applies the public output-bit toggles at each leaf. Computing the nontrivial prefix conjunctions costs

$$L(k) = 2^k - 2 \quad (14)$$

Toffolis. Measured erasure with phase correction clears each selector. The placed lookup uses at most $512 + 2k$ qubits, which fits the available initialization workspace for $k \leq 128$. The supported five-bit prefixes have $m \leq 25$ and hence $k \leq 126$.

Each of the $L(k)$ internal lookup measurements contributes the common branch-amplitude factor $2^{-1/2}$. Every subsequent point translation has 512 internal measurements with the same property. There are $103 - m$ translations after lookup, so the internal measurement count is

$$M = L(k) + 512(103 - m). \quad (15)$$

For each scalar-result pair, the program has 2^M internal histories, each with terminal state $2^{-M/2}$ times the corresponding analytic branch state. Thus their probabilities sum to the analytic branch probability: $2^M(2^{-M/2})^2 = 1$. The proof derives both the history multiplicity and common amplitude from the generated program. This establishes the distribution equality used in Theorem 1 despite the lookup’s additional measurements.

The lookup contains the only initial-prefix Toffolis. The remaining translations comprise one binary step and $102 - m$ five-bit steps. Adding their exact public-table-dependent counts defines $C(Q, m)$. Applying the uniform bound P_5 gives (1). Classical control selects only zero-cost alternatives in the Toffoli metric, so the count is path-independent. Normalization of the branch probabilities then proves its equality with the expectation.

6.2 Prefix-width tradeoffs

The initial lookup’s CNOT count is at most $514 \cdot 2^k - 2$. The packed table stores 2^k point codes of 512 bits each, or $64 \cdot 2^k$ bytes. Table 2 compares prefixes near the minimum of the five-bit Toffoli bound.

Prefix bits	Program Toffoli bound	Lookup CNOT bound	Packed table
11	37,642,109,902	1,052,670	128 KiB
16	37,269,475,486	33,685,502	4 MiB
21	36,898,809,198	1,077,936,126	128 MiB
26	36,591,123,006	34,493,956,094	4 GiB
31	38,298,799,886	1,103,806,595,070	128 GiB

Table 2: Verified bounds for the five-bit initial-lookup family. CNOT accounting covers the initial lookup. Storage counts packed public point codes.

The difference between consecutive bounds is

$$B_5(m+1) - B_5(m) = 31 \cdot 2^{1+5m} - 372,697,904.$$

It is negative through $m = 4$ and positive from $m = 5$ onward. This proves the minimum at the 26-bit prefix. The formal result also checks the inequality over the finite supported range. The minimum concerns this uniform upper-bound formula. Selecting the best table for an individual public point, or minimizing a combined Toffoli, CNOT, and memory cost, requires a different objective.

Initial lookup also applies to the earlier scalar-window constructions. For an even prefix $2 \leq k \leq 128$ in the two-bit construction, its upper bound is

$$B_2(k) = (256 - k/2)P_2 + 2^k - 2.$$

For a prefix $k = 1 + 3m$, $0 \leq m \leq 42$, in the three-bit construction it is

$$B_3(m) = P_1 + (170 - m)P_3 + 2^{1+3m} - 2.$$

A checked 20-bit two-bit prefix gives 91,603,405,850 Toffolis with a 64 MiB packed table. A checked 28-bit three-bit prefix gives 60,605,368,096 with a 16 GiB table. The same symbolic lookup and branch-multiplicity proofs support these constructions.

7 Lean proof decomposition

The current execution, recovery, pathwise-cost, and expected-cost theorems pass Lean 4.34.0-rc2. Their transitive axiom sets contain propositional extensionality, classical choice, and quotient soundness. The former native-decision dependencies for finite layout and gate-count calculations have been replaced by kernel-checked results. The formal source revision for the completed constructions is `ed35b6e8b517e77a9b76a0016e4353a8805d0e51`.

Structural count lemmas reduce generated-circuit counting to component arithmetic. The proofs establish count behavior under concatenation, reversal, index placement, and recursive generation, and then substitute the fixed widths and round count. For the shared endpoint writer, an explicit prefix-tree constructor admits an inductive action and count proof. Evaluating a general gate-cancellation transformation on the full writer had required expansion of its generated gate list. The structured constructor reduces that obligation to the prefix recurrence and leaf actions in Section 2.

Small layout lemmas prove bounds and disjointness for the specific fields that a component accesses. Composition uses these checked lemmas while keeping arithmetic definitions folded. Explicit normalization of field widths and natural-number powers reduces definitional conversion at component boundaries. Closed integer calculations are left until structural formulas have reduced their inputs. These changes preserve the algorithms and theorem hypotheses.

The coherent translation theorem takes an arbitrary finite table of valid public addends. Its common-amplitude conclusion supports every address superposition. Scalar-loop proofs separately establish label enumeration, group value, Fourier phase, and workspace state, then compose those results. Finite-range partition identities keep scalar enumeration symbolic at size 2^{256} . The history proof similarly uses symbolic cardinalities and powers in (15). The analytic recovery theorem applies once the resulting marginal equals (5).

This decomposition permits reuse of checked arithmetic, lookup, and Fourier results across window widths. It reduces repeated elaboration and the finite expressions that the kernel must evaluate.

8 Resource limits and further reductions

The bounds concern exact logical operations and a fixed logical allocation. Approximate rotation synthesis, error correction, routing, and fault-tolerant gate factories require additional resource models. The complete program’s CNOT count and depth also remain to be proved. Table 2 bounds only lookup CNOTs, and its packed-table sizes exclude the storage required to materialize the generated circuit representation. The proofs treat the public table and circuit symbolically. The decoder bounds count calls and candidate tests. They leave classical scalar-multiplication and field-arithmetic bit costs to further analysis.

The 26-bit prefix minimizes the proved Toffoli formula within the supported five-bit family. The 21-bit prefix uses a smaller public table and fewer lookup CNOTs at a higher Toffoli bound, so both constructions remain relevant to resource comparisons. Quantum-width minimality and global Toffoli optimality remain open.

A further endpoint proposal shares low-bit comparisons between successive remainder positions. It retains five high bits and traverses four low bits, using three prefix-scratch bits for each endpoint. A full four-bit traversal costs 30 Toffolis, and a three-position tail costs 9. The proposed complete remainder-pass count is

$$4(16 \cdot 30 + 9) + 2 \cdot 17 \cdot 36 + 7 \cdot 259 = 4,993,$$

compared with (2). This is a construction estimate. Interleaving the paired traversals with carry updates, reversing them, handling the partial block, and proving the compact placement remain obligations. The checked bounds in this report use the 9,897-Toffoli pass.

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