

Windowed Euclidean Arithmetic in an 833-Qubit secp256k1 ECDLP Program

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Abstract

We verify a secp256k1 elliptic-curve discrete-logarithm program with a fixed allocation of 833 logical qubits and exactly 588,551,462,912 Toffoli-class operations per execution. The program shares a temporary qubit between field-arithmetic and point-comparison stages. Round-indexed bounds on Euclidean coefficients shorten reversible endpoint searches, saving 192,739,491,840 Toffoli-class operations per complete execution. Lean 4 checks the generated arithmetic, workspace restoration, complete point translation, both semiclassical scalar loops, and the analytic recovery distribution. For every valid nonzero secp256k1 public key, 26 independent executions recover the private scalar with probability greater than 99 percent. The exact pathwise gate count also equals its expectation under the program’s semantic branch probabilities. The allocated width gives a peak-live upper bound.

1 Complete-program result

The program implements the complete double-scalar computation used by the semiclassical Shor algorithm for the elliptic-curve discrete-logarithm problem on secp256k1 [3, 2]. For the standard generator G , let q denote its prime order. For a valid public point $Q = [d]G$ with $0 < d < q$, the branch-state expansion after both scalar loops ranges over $a, b \in \{0, \dots, 2^{256} - 1\}$, with point values

$$[a]G + [b]Q = [a + db]G.$$

The loops recycle one control qubit, measuring and clearing it after each controlled translation and retaining the Fourier results in classical storage. The point register starts at the identity, and the arithmetic workspace and control start clear. The point operation covers the identity, ordinary addition, inverse pairs, and doubling. Its arithmetic workspace returns to its specified input state on every measurement branch. Summation over the internal measurement records gives the analytic scalar-pair distribution used by the recovery proof.

Set $N = 2^{256}$, and for $0 \leq t < q$ define the nearest Fourier bin

$$J_N(t) = \left\lfloor \frac{2Nt + q}{2q} \right\rfloor.$$

The notation $[x]_q$ denotes the least nonnegative residue of x modulo q . For every $0 < t < q$, the selected scalar pair is

$$(J_N(t), J_N([dt]_q)).$$

The checked decoder maps every selected pair to d , and the one-run probability of the selected family is at least

$$\frac{q-1}{q} \frac{2401}{14641}.$$

The independent-product theorem proves that 26 executions contain at least one selected outcome with probability greater than 99/100, and every selected outcome recovers d . If p_{sel} denotes the one-run mass of this family, the repeated-run probability is $1 - (1 - p_{\text{sel}})^{26}$. A mathematical decoder and public-key check establish recovery. The repetition theorem uses the independent product of the one-run law. Compiling repetition and classical decoding into one executable controller remains separate work.

VQ is a Lean 4 framework for exact functional verification and logical resource analysis of quantum programs [5, 1]. It represents unitary circuits, reversible basis permutations, and hybrid programs with measurement, reset, classical storage, and feed-forward. Its exact amplitude semantics retain each measurement branch. Functional proofs relate generated programs to mathematical specifications, while resource functions calculate width and path-dependent gate counts from the same program syntax. The secp256k1 development uses Mathlib for finite sums, complex analysis, modular arithmetic, and elliptic-curve groups [6].

2 Shared auxiliary qubit

Sharing logical qubit 832, using zero-based indexing, between the zero-factor and equality roles reduces the complete program’s fixed allocation from 834 to 833 logical qubits. The 834-qubit allocation assigned separate logical qubits to the two temporary Boolean values [4]. Totalized field division and multiplication use a zero-factor tag to replace a zero factor by one during a reversible arithmetic stage. Point negation and exceptional-case correction use an equality temporary while computing and erasing comparison results.

Retained division preserves its denominator in the input register and restores the zero-factor tag. Retained multiplication preserves one factor in the input register and restores the same tag.

The program uses the shared qubit in this order:

1. use the equality temporary to compute the four exceptional-input tags, clearing the temporary after each comparison;
2. compute the zero-factor tag, perform retained division, and clear the tag;
3. compute the zero-factor tag, perform retained multiplication, and clear the tag;
4. use the equality temporary for controlled negation and clear it;
5. apply the exceptional-case correction, then use the equality temporary to erase the four exceptional tags.

The arithmetic proofs establish that the zero-factor tag is clear after division and multiplication. The comparison proofs establish that the equality temporary is clear after input tagging, negation, and exceptional-tag erasure. The composed translation proof uses these restoration results at the role boundaries. The shared allocation therefore preserves the point-translation function and gate multiplicities while changing the target index of operations on the shared qubit.

Three 256-qubit field registers use 768 qubits. The recycled Fourier control uses one. Six bits extend two Euclidean work registers from 256 to 259 bits. Four store phase, iteration, and sign control states. Thirty-five encode three operand lengths and the current shift amount. One arithmetic extension bit and a 13-bit pool provide temporary storage reused by the Euclidean substeps. These packed Euclidean fields use $6+4+35+14 = 59$ qubits. Four exceptional-case tags and the shared auxiliary qubit complete the allocation:

$$768 + 1 + 59 + 4 + 1 = 833.$$

Lean evaluates this layout width and proves that every gate and measurement references a valid location. This well-formedness result covers the complete two-loop program for every compiled public-point value.

3 Windowed Euclidean arithmetic

The field arithmetic uses a fixed 1,620-round packed binary-Euclidean computation. At a work-register exchange, reversible searches recover the operand endpoints needed by later rounds. The full-width implementation searches all possible positions. The revised circuit restricts each search to a prefix or suffix whose endpoints depend on the emitted round number. Both constructions use the packed arithmetic described in the preceding report [4].

For a nonempty completed quotient prefix $u_1, \dots, u_m$, define $C_{-1} = 0$, $C_0 = 1$, and $C_j = u_j C_{j-1} + C_{j-2}$. Let w be the sum of the quotient bit lengths and $C = C_m$. The first quotient is at least two, and the later quotients are positive. Lean proves

$$C \geq \eta \lambda^w, \quad \lambda = (2 + \sqrt{3})^{1/3}, \quad \eta = \frac{4\sqrt{3} - 3}{13} \lambda^2.$$

An upper estimate on the sum of consecutive coefficients supplies a complementary remainder bound. A quotient-prefix invariant identifies these coefficients in every reachable Euclidean phase and relates the accumulated weight to the round index. The proof includes quotient completion, partial shifts, register exchange, and the terminal remainder.

The circuit generator uses integer arithmetic to calculate the endpoints. Write $r = w \bmod 49$ and $s = r \bmod 19$, and define

$$B(w) = \max\left(1, 31 \left\lfloor \frac{w}{49} \right\rfloor + 12 \left\lfloor \frac{r}{19} \right\rfloor + 5 \left\lfloor \frac{s}{8} \right\rfloor\right).$$

The proved inequalities

$$\eta > \frac{1}{2}, \quad \lambda^{49} > 2^{31}, \quad \lambda^{19} > 2^{12}, \quad \lambda^8 > 2^5$$

make $B(w)$ a lower bound on the coefficient bit length. For one-based emitted round T , set $w = \lfloor T/4 \rfloor$. The generated upper search uses a prefix of width $\min(w + 3, 258) + 1$. The lower search uses the suffix starting at zero-based source index $B(w) + 1$, of width $258 - B(w)$, through source index 258. The reachable-state proof places every selected endpoint inside these intervals. The lower-search proof also covers the encoded zero result.

Each shortened search writes the same endpoint as its full-width counterpart and restores its temporary storage. Composing the searches proves the complete round action. Separate inactive and terminal-state theorems cover rounds that perform no Euclidean update, including wrap of the terminal counter's low word. The indexed schedule therefore reaches the required terminal encoding for every admissible input. Inverse extraction followed by reversal returns a reduced inverse and restores the prepared input. Reversing input preparation gives the field caller's required workspace state.

Retained division and multiplication use this forward computation and its reverse. Their measured cleanup and phase correction preserve the branch amplitudes required for coherent point translation. The complete translation proof then supplies the scalar-loop and recovery proofs with the changed arithmetic. The round count and allocation remain fixed.

4 Logical resources

The Toffoli metric counts one compiled CCZ for each reversible controlled-controlled- X operation, which the compiler represents as a CCZ conjugated by Hadamard gates on its target. We call each counted CCZ a Toffoli-class operation. Measurements, classically selected Pauli operations, and phase rotations contribute zero to this metric. The count describes compiled reversible Toffoli-class operations and excludes error correction, routing, and physical gate synthesis.

The fixed part of each Euclidean round contributes 77,405 Toffoli-class operations. The shortened endpoint-search blocks and their surrounding control logic contribute 141,696,120 across the schedule. Input preparation contributes 35,204. The prepared forward schedule has count

$$S = 35,204 + 1,620 \cdot 77,405 + 141,696,120 = 267,127,424.$$

Lean derives this equality from the emitted gate lists and composes the following counts from the same constructors.

Operation	Composition	Toffoli-class count
Prepared forward schedule	S	267,127,424
Retained division	$2S + 30,323,778$	564,578,626
Retained multiplication	$2S + 30,323,778$	564,578,626
Complete controlled point translation	$P = 4S + 81,004,880$	1,149,514,576
Complete ECDLP execution	$512P$	588,551,462,912

The full-width prepared schedule costs 361,238,504, so the shortened searches save 94,111,080 per forward schedule and 192,739,491,840 per complete ECDLP execution. The complete-program syntax has the singleton Toffoli range $\{588,551,462,912\}$. Every execution path therefore has this count. A separate expectation theorem uses the program's normalized semantic branch probabilities to prove

$$\mathbb{E}[\text{Toffoli-class operations}] = 588,551,462,912.$$

The expectation theorem covers every basis input within the allocated width and every initial classical input. Its phase representation must support the exact rotations used by the scalar loops. The recovery theorem further requires the initialized algorithm state and valid public key specified above.

The exact width theorem concerns the program’s declared quantum allocation. Because the program allocates one fixed register array, 833 also bounds its peak live logical qubits. Width minimality remains open and would require a lower bound over an explicit class of implementations.

5 Formal evidence and limits

The results use repository revision 2f80554, Lean 4.32.0, and the pinned Mathlib dependencies. The checked chain contains the shared layout, component placement, coefficient-window bounds, indexed Euclidean rounds, extraction and reverse cleanup, basis-state semantics for each changed stage, complete point translation, two-scalar semantics, scalar-pair marginal, selected-recovery bounds, complete-program well-formedness, exact width, and exact pathwise and expected Toffoli counts. The semantic, probability, and resource statements refer to the same generated operation list.

The proof dependency set includes propositional extensionality, classical choice, quotient soundness, and private axioms generated by Lean’s native decision procedure for fixed finite layout and gate-list computations. The semantic and resource theorems describe the exact logical program model. A physical resource estimate would require a fault-tolerant architecture, routing, phase synthesis, code parameters, and factory allocation.

Further reductions may come from shortening the coefficient-update passes or changing the point-addition arithmetic. The coefficient bounds already give a cost model for shorter update passes, but the corresponding generated circuit and its semantic proof remain to be constructed. Such a change must retain the proved reverse cleanup and coherent measurement behavior before its projected savings can enter the complete-program resource theorem.

References

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