

VQ: Exact Functional Verification and Certified Logical-Resource Analysis of Quantum Algorithms

Jamie Stephens
js@morphism.com

26 August 2026

Abstract

VQ is a Lean 4 framework for defining quantum circuits and hybrid quantum–classical programs, proving their behavior, and deriving logical resource results from the same executable syntax. Its semantic models cover well-formed finite circuits over exact dyadic cyclotomic amplitudes, reversible basis permutations, and programs with measurement, reset, immutable classical input, mutable storage, and feed-forward. Functional specifications state the admitted inputs, output values, preserved registers, workspace cleanup, and the input independence required for measured components to act coherently on superpositions. Static analyses report declared width, touched wires, gate classes, dependency depth, measurement, reset, and classical control, while semantic branch weights support expected additive path costs for basis inputs under finite input laws. Checked applications include parameterized arithmetic, measurement-assisted addition and lookup cleanup, secp256k1 inversion and point addition. The repository’s test target also checks exact quantum Fourier transform (QFT), phase-estimation, and secp256k1 elliptic-curve discrete-logarithm problem (ECDLP) developments. VQ certifies ideal finite semantics and logical resources. Its reusable library API excludes density operators, physical noise, routing transformations, fault-tolerant costs, and useful-precision phase synthesis connected to terminal algorithm error bounds.

1 Generated-program semantics and resources

VQ treats a quantum algorithm as an executable Lean value accompanied by theorems about that value. A generator may depend on widths, moduli, layouts, or classical data, and a closed artifact fixes those parameters. The application results in this report name the generated value or connect source and target values through a proved translation [16].

VQ states functional behavior, preservation, cleanup, coherence, terminal success, and resources through explicit theorem statements. In the checked applications, semantic and resource theorems name the same generated syntax. When an application theorem names both results, Lean checks their shared program expression.

Lean defines the generators, mathematical objects, and proofs [3]. The kernel checks their proof terms against pinned Lean, Mathlib, and supporting library revisions [17]. The analyzed artifact is revision `d407e8c`. Its top-level build checks the reusable libraries, developments imported by the test target, manifest verifier, and specialized runtime point-adder package.

The repository assigns evidence status according to API reuse and theorem quantification. Every VQ result called checked in this report is a kernel-checked theorem at the pinned revision. A closed artifact fixes its parameters and may have any of the statuses in Table 1.

Evidence status	Meaning
Production	A reusable declaration imported by a top-level reusable-library target.
Test-local	A kernel-checked declaration imported by the repository’s top-level test target, pending review of its public API.
Example-specific	A kernel-checked result specialized to one program, modulus, width, layout, or example package.

Table 1: Evidence status used for VQ results in this report.

2 Exact semantics for circuits, reversible arithmetic, and hybrid programs

The unitary circuit language records a quantum width and an ordered list of primitive gates. Its gate set contains Hadamard, X, Y, Z, S, inverse S, T, inverse T, exact higher dyadic phases and their inverses, CNOT, and CCZ. Well-formedness checks wire bounds, distinct operands, and whether the selected phase level contains every root of unity used by the circuit.

At semantic level $\ell \geq 3$, amplitudes lie in the dyadic cyclotomic ring

$$D_\ell = \left(\mathbb{Z}[X]/(X^{2^{\ell-1}} + 1) \right) [1/2].$$

The class of X represents a primitive 2^ℓ th root of unity, and the ring contains $1/\sqrt{2}$. Circuit execution acts on a basis-indexed amplitude function, and the library proves linearity, well-formed support, composition, and exact denotation. The model compares exact algebraic amplitudes.

Reversible circuits use X, CNOT, and Toffoli gates and denote transformations of basis indices. The library proves that every well-formed reversible circuit acts injectively within its declared width. A verified compiler maps this syntax into the unitary language, replacing Toffoli by a Hadamard–CCZ–Hadamard sequence and proving the same basis action in the exact semantics.

Hybrid programs add measurement, reset, immutable input reads, mutable classical bits, bit inversion, and classical branches. Execution produces subnormalized quantum branches carrying their measurement record and classical state. For a well-formed program acting on a width-supported unit-norm exact state, a production theorem proves that the branch weights sum to one.

Program form	Semantic object	Principal claims
Unitary circuit	Exact linear action on pure-state amplitudes	State equality, basis-input full-register and selected-wire probability, test-local semantic adjoint laws, and gate-level resources
Reversible circuit	Injective permutation of bounded basis indices	Arithmetic action, preserved fields, cleanup, and compilation to unitary gates
Hybrid program	Finite list of subnormalized branches with records and classical state	Measurement-assisted behavior, reset, feed-forward, normalization, and branch-sensitive cost

Table 2: VQ syntax classes and their semantic roles.

Integer and bit transformations establish reversible arithmetic before verified compilation. Measured uncomputation retains its record and branch structure. Interference arguments use the exact pure-state linear semantics.

3 Value, register, cleanup, and coherence specifications

A component specification fixes its register width, admitted basis inputs, and acceptable outputs. The output relation covers the complete basis index, so a single theorem can state the arithmetic result, preserved source fields, unchanged control bits, and zeroed workspace. Layout and placement theorems preserve a local correctness theorem when embedding the component into disjoint fields of a larger register.

Measurement-assisted components require an additional condition. For fixed immutable classical input x and admitted basis input i , let $\mathcal{B}(P, x, i)$ denote the terminal branch list of program P . Write $|\psi_b\rangle$ for branch b 's quantum state and $r(b)$ for its measurement record. The proof identifies an output index $f_x(i)$ and amplitudes $\alpha_{x,r}$ indexed by measurement records such that every $b \in \mathcal{B}(P, x, i)$ satisfies

$$|\psi_b\rangle = \alpha_{x,r(b)} |f_x(i)\rangle, \quad \sum_{b \in \mathcal{B}(P, x, i)} \|\psi_b\|^2 = 1.$$

For fixed x , the same $\alpha_{x,r}$ applies to every admitted i . By linearity, every terminal branch with record r maps an admitted finite superposition to the corresponding output superposition multiplied by $\alpha_{x,r}$. Measurement-record probabilities are therefore independent of the admitted basis input while x remains fixed.

Terminal algorithms use event probabilities over exposed outcomes. Circuit probabilities and selected-wire marginals are production functions for a computational-basis input, with exact equalities at every admitted phase level and ordered rational bounds for the level-three model. Test-local hybrid theorems prove that terminal-event probability is unchanged when a branch list is reordered or one branch is replaced by branches with the same total weight.

The runtime point adder adds an immutable `secp256k1` affine offset to a quantum affine target. Its addable domain requires both inputs to be reduced points on the curve with distinct x-coordinates. For this component, a production certificate takes a probability distribution on a finite input type, a designated exceptional subset, and a rational upper bound on the subset’s probability. It proves exact behavior outside that subset and bounds its probability under the stated distribution. The associated coherence theorem covers target superpositions that share one fixed immutable offset, while a quantum-state error theorem requires an additional relation between the distribution and the input amplitudes.

The production approximation library defines distances on finite complex vectors and matrices. Its terminal-vector relation identifies normalized vectors that differ by multiplication by one complex scalar of norm one. Applications supply normalization and unitarity premises. Test-local theorems derive event-probability loss bounds from operator error and compose omitted phase rotations through quantum Fourier and phase-estimation circuits. Test-local results include checked single-qubit phase-synthesis certificates, a three-qubit synthesized-QFT construction with total synthesis-error budget $\pi/4$, and a phase-estimation terminal-event probability bound derived from an explicit QFT error. Useful-precision synthesis and a production theorem connecting generated circuits to complex operators remain open.

4 Symbolic proofs at cryptographic widths

Direct execution reduces a finite program and gives exact results for small circuits, counterexamples, and reference calculations. Its cost grows with the materialized state dimension, the measurement branch count, and the degree of the cyclotomic ring. Cryptographic-width arithmetic uses symbolic theorems whose checking cost depends on circuit structure and arithmetic side conditions.

For reversible arithmetic, a proof follows the generator’s structure and shows how each gate changes selected bits or integer fields. The modular-arithmetic proof uses range lemmas, congruences, and invariants, while cleanup follows from explicit reverse computation or a component-specific measured-uncomputation theorem. The compiler theorem then transfers the basis action to exact quantum semantics.

Interference proofs use finite sums and character identities. The checked QFT, phase-estimation, and elliptic-curve discrete-logarithm theorems quantify over generated widths, basis labels, phase exponents, and selected outcomes. Finite-sum lemmas reduce their terminal distributions to algebraic identities.

Resource proofs follow the recursive circuit generator. Append, reverse, and fixed repetition have production reversible gate-count certificate laws. Placement has separate production count-preservation lemmas. Algorithm-level serial repetition theorems, including the ECDLP specialization, remain test-local.

5 Syntax-derived logical-resource theorems

Circuit resource functions report the declared width and compute touched wires, total gates, gate classes, and dependency depth. Depth uses a greedy per-wire logical schedule in which a gate waits for its operand wires and disjoint gates may run in parallel. The functions compute exact values for the supplied circuit. Application theorems use exact values or closed upper bounds that Lean reduces and compares.

Program resources include quantum width, immutable input width, mutable classical storage, gate classes, measurements, resets, classical operations, and quantum and classical dependency depths. A branch in the syntax produces a range with lower and upper endpoints by combining its arms. These endpoints may include an unreachable arm because the static analysis does not prove reachability.

Execution-cost semantics associates each terminal branch with the additive cost accumulated along its path. For in-range computational-basis inputs to a well-formed program, exact branch weights and a probability distribution over the finite input type define expected costs, probabilities of exceeding a cost threshold, static upper bounds, and Markov-inequality bounds. The theory covers additive path metrics such as executed CCZ count, but defines neither expected dependency depth nor expected stopping time.

Resource certificates compose bounds for generated reversible circuits, compiled circuits, and programs formed by embedding those circuits. A resource certificate proves an inequality about syntax, while a functional theorem

proves behavior. Combined application results name the same generated value in their functional and resource theorems.

Metric	Mathematical meaning	Result form	Outside the result
Quantum width	Declared whole-program register width	Exact syntax value or proved bound	Dynamic allocation and minimum-live-width analysis
Gate classes	Occurrences in the declared primitive basis	Exact syntax value or proved bound	Native hardware gates and synthesis not named by the model
Dependency depth	Greedy logical schedule over operand wires	Exact circuit value or program range	Routing, latency, factories, and physical concurrency
Hybrid operations	Measurement, reset, storage, and classical-control syntax	Exact values or branch ranges	Reachability-sensitive optimization and wall-clock timing
Expected additive cost	Semantic branch-weighted path cost under a finite input distribution	Exact expectation, upper bound, or threshold-probability bound	Expected depth and stochastic stopping policy
Coupling check	Each CNOT uses one graph edge, and each CCZ uses all three pairwise edges	Decidable conformance result	A verified routing transformation or native-device implementation

Table 3: Meaning and boundary of VQ resource results.

The reversible compiler represents each Toffoli by one CCZ and two Hadamards. Application reports label this count “CCZ/Toffoli-equivalent” when comparing the source arithmetic with the compiled circuit. A separate expansion charges seven T gates per CCZ. Gidney’s temporary logical AND uses four T gates for computation and none for erasure, and representing that construction requires a separate VQ semantic and resource theorem [4].

6 Project-build and verifier theorem packages

The repository build checks the source modules that state its results. The kernel checks declarations against the pinned source and compiled dependencies. Axiom audits on the runtime point-adder correctness theorem and the verifier’s typed claim constructors report exactly proposition extensionality, classical choice, and quotient soundness.

The submission verifier accepts a Lean module and a typed manifest that lists the required claims. Trusted code constructs each proposition from the manifest, and the submitted theorem must have that exact type. The audit rejects unsafe or partial declarations, restricts reachable axioms, and invokes a separate Lean kernel replay for the submitted modules.

The claim schema determines which propositions an external package may submit. Its circuit semantic claims cover specified basis-input columns and signed permutation entries, while its probability claims cover exact certainty or impossibility and level-three rational bounds. Reversible arithmetic has separate claim forms, as do width, used-wire, gate-class, expanded-gate, and depth metrics for circuits and programs. Branch-indexed runtime point-addition correctness, terminal hybrid-program events, and the ECDLP semantic and probability results remain in-repository because the claim schema cannot express them.

The trusted boundary includes the Lean kernel, the pinned Lean and Mathlib revisions, the pinned imported dependencies, VQ’s definitions, and the verifier implementation that parses manifests, prepares a temporary Lean project, constructs obligations, audits declarations, and runs kernel replay. Replay checks the submitted modules while imported VQ and dependency modules remain trusted compiled artifacts. Submitted Lean source executes with the invoking user’s filesystem and network authority.

7 Applications combining semantics, probability, and resources

VQ applies its semantic and resource models to reusable arithmetic families, closed cryptographic programs, and terminal algorithm endpoints. The applications retain distinct evidence status. Their numerical bounds use the

logical resource models defined in Table 3. In the ECDLP results, G denotes the standard secp256k1 generator and q denotes its subgroup order [15].

Let $N = 2^{256}$. An output pair (j_1, j_2) decodes to $k = D_N(j_1)$ and $v = D_N(j_2)$, where $D_N(j) = \lfloor (2qj + N)/(2N) \rfloor$. The recovery procedure rejects $k = 0$. Otherwise, it forms $e = vk^{-1} \bmod q$ and accepts exactly when $e < q$ and $eG = Q$. The acceptance probability is the total terminal probability of pairs that pass this procedure, and the soundness theorem proves that every accepted e equals d when $Q = dG$ and $d < q$.

The probability lower bound uses a subset of the accepted pairs indexed by $1 \leq k < q$. Define $R_N(t) = \lfloor (2Nt + q)/(2q) \rfloor$ for $0 \leq t < q$. The subset consists of the pairs $(R_N(k), R_N(dk \bmod q))$, each of which decodes to k and $dk \bmod q$ and therefore recovers d .

The window-3 oracle represents each fixed base by the doubling table $P, 2P, \dots, 2^{255}P$ and processes successive scalar bits in groups of three from the low end. For each complete group, its three-bit value selects the corresponding subset sum of three table points, the circuit adds that point to the accumulator, and the circuit uncomputes the lookup. The final high bit uses one controlled addition, and the oracle applies the construction independently to G and Q before the two scalar-register QFTs.

A controlled phase at index r in the QFT has angle $2\pi/2^r$. Cutoff c retains exactly the controlled phases with $r \leq c$, together with every Hadamard and all final swaps. The cutoff-17 ECDLP construction applies this rule to each 256-wire scalar-register QFT.

Table 4: Representative checked applications at revision d407e8c.

Application	Semantic, probability, or approximation result	Logical-resource result	Evidence status
Reversible arithmetic	Parameterized wrapped and modular addition, subtraction, negation, multiplication, squaring, inversion components, and placement into larger layouts	Symbolic width, gate-class, and depth formulas for selected generators	Production families and example-specific constructions
Measured addition and quantum read-only memory (QROM) cleanup	Addition and quantum-indexed lookup with input-independent measurement-record amplitudes, exact cleanup, and coherent finite-superposition action	Width, gates, measurements, resets, feed-forward, storage, and dependency-depth bounds	Production semantic interfaces and example-specific constructions
secp256k1 Euclidean field-inversion circuit	Totalized field inversion for every reduced input, returning zero on zero and the multiplicative inverse on every nonzero input, with source preservation and workspace cleanup after a proved fixed Euclidean schedule	Fixed logical width 860 and at most 2,162,792,474 CCZ/Toffoli-equivalent gates	Example-specific
secp256k1 point addition by a classical runtime offset	Exact affine addition when both inputs are reduced on-curve points with distinct x-coordinates, cleanup on every representable clear-workspace input, and coherent action for each fixed runtime offset	Fixed logical width 12,317 and at most 147,228,788 CCZ/Toffoli-equivalent gates	Example-specific
Totalized secp256k1 group law and scalar multiplication	The production mathematical model represents infinity by $(0, 0)$ and defines addition on every encoded input. Test-local theorems cover coherent point addition and fixed-base and two-scalar circuit action on exact supported superpositions	Syntax-derived fixed-width gate-class and depth bounds	Production mathematical API and test-local circuit families

Application	Semantic, probability, or approximation result	Logical-resource result	Evidence status
QFT, phase estimation, and order finding	Exact QFT basis action, exact phase estimation for eigenphases $a/2^m$ representable by the m -qubit phase register, and an $N = 15$, base-two order-finding instance with exact factor probability $3/4$	Exact family formulas and closed gate-class results	Test-local and example-specific
secp256k1 ECDLP	At level $\ell \geq 257$, for a valid encoded public point $Q = dG$ with $d < q$, the window-3 endpoint has an exact terminal distribution and one-run acceptance probability at least $((q - 1)/q)(2401/14641)$. Every accepted pair recovers d , and 26 independent runs succeed with probability above 99 percent	One-run logical width 12,829 and at most 25,324,153,616 CCZ/Toffoli-equivalent gates. The 26 sequential copies retain width 12,829 and use at most 658,427,994,016 such gates	Test-local
Approximate QFT and phase estimation	Operator-error composition, event-probability bounds, and an omitted-exact-phase ECDLP reference bound with one-run probability at least $1/32$ for the selected subset of scalar pairs defined above, for the 12,831-wire circuit, level $\ell \geq 257$, a valid encoded $Q = dG$, and the cutoff-17 rule in both scalar-register QFTs	Exact formulas for omitted phase gates and the retained circuit syntax	Test-local

The runtime point-adder accepts two 256-bit affine coordinates through immutable classical input and acts on a 512-qubit affine target over secp256k1 [15]. Its value theorem uses the addable domain defined above, while its cleanup theorem covers every clear-workspace target and offset whose coordinates are below the field prime, including off-curve pairs and inputs beyond the affine value clause. The program has zero measurements, resets, and mutable classical bits, and its resource theorem concerns the same closed syntax as its correctness theorem.

The ECDLP theorem proves the terminal scalar-pair distribution and secret-recovery map for a Shor-style secp256k1 circuit [14, 12]. It derives the acceptance-probability lower bound from exact finite sums and analytic inequalities over every admitted public point and secret. The serial resource theorem reports the one-run circuit width and multiplies gate-list totals by 26.

The ECDLP resource result treats exact positive-sign QFT phase gates as logical primitives. Measurement, reset, classical decoding, public verification, routing, error correction, and phase synthesis with an algorithm-level error bound lie outside the stated serial gate bound. The fixed width states a whole-program register extent, and VQ has no minimum peak-live storage theorem.

Checked placement rules reuse parameterized components in larger register layouts. Artifact-level theorems state functional, probability, approximation, and logical-resource properties of one closed construction. The manifest verifier checks supplied Lean proofs against a consumer-supplied manifest within VQ’s fixed typed claim schema.

8 Comparison of semantic, proof, and resource capabilities

VQ applies when an analysis requires exact functional and probability results, explicit cleanup or coherence conditions, and logical-resource theorems about one finite circuit or hybrid program. Selected checked applications state semantic and resource theorems about the same executable Lean syntax. The manifest verifier checks third-party theorems against a fixed claim schema and audits the axioms reachable from each submitted theorem. Table 5 records the evidence mechanism for each capability. An entry marked “Unreported” means that the cited source does not report the capability.

Table 5: Capabilities of VQ and selected quantum verification and resource-analysis systems.

System	Functional evidence	Measurement and state model	Executable artifact or transformation	Logical resources	Expected resources	Physical resources
VQ	Lean proofs of exact behavior, cleanup, coherence, and basis-input circuit probabilities [16]	Finite pure-state branches with measurement, reset, and feed-forward	Register placement and executable Lean syntax	Lean theorems for width, wire use, gate classes, and dependency depth	Lean expected-cost and tail-bound theorems under finite input distributions	No verified physical-resource model
SQIR/VOQC	Coq proofs of transformations and algorithm results [6, 11]	Unitary, density-matrix, and nondeterministic semantics	Verified optimization, gate conversion, mapping, and an extracted optimizer	Checked Shor bound in the extraction gate set	Unreported	Unreported
Qbricks	Why3 obligations for parameterized circuit builders [1]	Output probabilities under terminal measurement	WhyML builders and OCaml extraction	Size, width, and ancillary-qubit postconditions	Unreported	Unreported
QWIRE	Coq circuit equations and examples [10, 7]	Density matrices and measurement	Typed circuits and OpenQASM output	Unreported	Unreported	Unreported
CoqQ	Coq partial- and total-correctness proofs [18]	Density operators, superoperators, classical control, and loops	Unreported	Unreported	Unreported	Unreported
Qafny	Dafny obligations with a Coq metatheory [8]	Typed states, measurement, conditionals, and loops	Tested Qafny-to-Dafny translation and a described SQIR/OpenQASM path	Unreported	Unreported	Unreported
QuRA	Unreported	Unreported	Proto-Quipper circuit builders and analyzer [2]	Solver-inferred parametric bounds	Unreported	Unreported
Qet	Unreported	Density matrices, mid-circuit measurement, and classical loops [9]	Expected-cost analyzer	Unreported	Z3 upper bounds for explicitly annotated operation costs	Unreported
Lean-QuantumAlg	Lean theorem endpoints under stated hypotheses [13]	Pure states, gates, and measurement	Executable Lean definitions	Lean resource endpoints from assumed query and iteration counts and formula estimates	Unreported	Unreported
Qualtran	Classical and tensor simulation with tests [5]	Unreported	Hierarchical operation decompositions and Cirq interoperability	Executable logical counts	Unreported	Surface-code estimates

The matrix distinguishes proof-assistant results, automated obligations, solver-backed analyses, and executable estimates. Entries described as Lean or Coq proofs denote proof terms checked by their kernels. Why3 and Dafny rely on automated back ends for generated obligations, satisfiability-modulo-theories (SMT) entries report inferred bounds, and Qualtran entries report Python analyses and tests.

The VQ row combines production APIs and checked application targets. Tables 1 and 4 state their status. QuRA and Qet provide paper soundness proofs for analyzers whose constraints go to SMT solvers. Runnable VOQC depends on Coq extraction, the OCaml toolchain and runtime, and its Python wrapper. The pinned QWIRE artifact identifies admitted compositionality lemmas used by downstream files. Qbricks uses documented axiomatic mathematical theories.

Selected specifications state cleanup, register preservation, measurement-record coherence, and finite-law expected additive costs. VQ covers measurement, reset, feed-forward, and expected path-cost theorems. Qbricks automates first-order obligations for static parameterized circuit builders.

9 Ideal finite semantics and logical-resource coverage

VQ certifies ideal finite semantics over declared logical syntax. Each theorem states the assumptions relevant to its claim, while application reports identify the input domain, gate basis, input law, and resource model where applicable. Table 6 records the remaining limits on those claims.

Table 6: Limits on VQ claims at the pinned revision.

Area	Present boundary	Consequence
Quantum states	Pure-state vectors and explicit ideal measurement branches	Density matrices, channels, physical noise, and open-system distance require separate density-operator semantics
Program syntax	Finite operation lists without loops or dynamic quantum allocation	Repetition and iterative algorithms require generated finite unrolling or a separate mathematical product model
Exact phases	Dyadic roots of unity selected by a finite semantic level	Arbitrary rotations require approximation and synthesis theorems
Approximation	Production state and operator relations. Circuit-to-complex execution, event-probability bounds, and phase-synthesis theorems remain test-local	Production integration of phase-synthesis error with terminal success remains open
Evaluation	Direct state and branch evaluation is exponential	Cryptographic-width proofs require handwritten symbolic semantics, invariants, sums, and resource recurrences
Compilation	Verified reversible-to-unitary compilation	General optimization, external assembly translation, and routing transformations remain unverified
Public API	Terminal hybrid observation, ECDLP, QFT, phase estimation, repetition, approximation transport, and phase synthesis remain under the top-level test target	Their theorems are kernel checked but do not yet define a reviewed production interface or manifest claim
Submission checking	Submitted Lean source runs with the invoking user’s authority	The verifier’s theorem claim assumes trusted execution of submission source

At revision `d407e8c`, VQ certifies exact ideal behavior or stated logical-resource bounds for artifacts covered by its theorems within explicit domains. Across separate applications, the repository establishes value semantics, preservation, cleanup, coherent measured execution, analytic probabilities, and syntax-derived resources. Claims about physical execution require additional verified models for synthesis, routing, fault tolerance, noise, and timing.

References

- [1] Christophe Charetton, Sébastien Bardin, François Bobot, Valentin Perrelle, and Benoît Valiron. An automated deductive verification framework for circuit-building quantum programs. In *Programming Languages and Systems: ESOP 2021*, volume 12648 of *Lecture Notes in Computer Science*, pages 148–177. Springer, 2021.
- [2] Andrea Colledan and Ugo Dal Lago. Flexible type-based resource estimation in quantum circuit description languages. *Proceedings of the ACM on Programming Languages*, 9(POPL):1386–1416, 2025.
- [3] Leonardo de Moura and Sebastian Ullrich. The Lean 4 theorem prover and programming language. In *Automated Deduction: CADE 28*, volume 12699 of *Lecture Notes in Computer Science*, pages 625–635. Springer, 2021.
- [4] Craig Gidney. Halving the cost of quantum addition. *Quantum*, 2:74, 2018.
- [5] Matthew P. Harrigan, Tanuj Khattar, Charles Yuan, Anurudh Peduri, Noureldin Yosri, Fionn D. Malone, Ryan Babbush, and Nicholas C. Rubin. Expressing and analyzing quantum algorithms with Qualtran. arXiv:2409.04643, 2024.

- [6] Kesha Hietala, Robert Rand, Shih-Han Hung, Xiaodi Wu, and Michael Hicks. A verified optimizer for quantum circuits. *Proceedings of the ACM on Programming Languages*, 5(POPL):37:1–37:29, 2021.
- [7] inQWIRE contributors. QWIRE software artifact. Coq software repository, 2025. Revision 6cb3e5acdf3cc7351ac839c2339f5124a160cc31, accessed 26 August 2026.
- [8] Liyi Li, Mingwei Zhu, Rance Cleaveland, Alexander Nicoletti, Yi Lee, Le Chang, and Xiaodi Wu. Qafny: A quantum-program verifier. In *38th European Conference on Object-Oriented Programming*, volume 313 of *Leibniz International Proceedings in Informatics*, pages 24:1–24:31, 2024.
- [9] Georg Moser and Michael Schaper. Automated expected cost analysis for quantum programs. arXiv:2604.03971, 2026.
- [10] Jennifer Paykin, Robert Rand, and Steve Zdancewic. QWIRE: A core language for quantum circuits. In *Proceedings of the 44th ACM SIGPLAN Symposium on Principles of Programming Languages*, pages 846–858. ACM, 2017.
- [11] Yuxiang Peng, Kesha Hietala, Runzhou Tao, Liyi Li, Robert Rand, Michael Hicks, and Xiaodi Wu. A formally certified end-to-end implementation of Shor’s factorization algorithm. *Proceedings of the National Academy of Sciences*, 120(21):e2218775120, 2023.
- [12] John Proos and Christof Zalka. Shor’s discrete logarithm quantum algorithm for elliptic curves. *Quantum Information and Computation*, 3(4):317–344, 2003.
- [13] QudeLeap contributors. Lean-QuantumAlg. Lean 4 software repository, 2026. Revision 7e80846034b9e76fdeded711775a513a7d5ba917, accessed 26 August 2026.
- [14] Peter W. Shor. Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer. *SIAM Journal on Computing*, 26(5):1484–1509, 1997.
- [15] Standards for Efficient Cryptography Group. SEC 2: Recommended elliptic curve domain parameters. Technical report, Certicom Research, 2010. Version 2.0.
- [16] Jamie Stephens. VQ: Exact functional verification and certified logical resources. Lean 4 software artifact, 2026. Revision d407e8c233788ab20f283c018e1e42fe9b494072, accessed 26 August 2026.
- [17] The mathlib Community. The Lean mathematical library. In *Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs*, pages 367–381. ACM, 2020.
- [18] Li Zhou, Gilles Barthe, Pierre-Yves Strub, Junyi Liu, and Mingsheng Ying. CoqQ: Foundational verification of quantum programs. *Proceedings of the ACM on Programming Languages*, 7(POPL):833–865, 2023.