

VQ: Quantum Algorithm Development and Verification with an Evolving Knowledge Base

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Abstract

VQ is a Lean-based system for developing and verifying quantum algorithms, with a current focus on elliptic-curve arithmetic and the elliptic-curve discrete logarithm problem. It connects algorithm constructions, formal proofs, resource analysis, and reusable research knowledge in one repository. Its two goals are to improve algorithms and to improve the knowledge used to develop them. The algorithm library provides exact semantics, specifications for reversible and measured components, composition theorems, and logical-resource accounting for the generated programs. It retains alternative constructions with their input, workspace, and preservation conditions. The knowledge base links declarations to applicability, methods, skills, references, and research questions. Experiment records supply evidence for revisions to both collections. Checked results include complete secp256k1 point addition, ECDLP recovery probability greater than 99 percent, and resource, lookup, decoder, and Euclidean schedule bounds. A comparison of squaring components illustrates the connection between the formal library and reusable knowledge. The report compares VQ with systems for quantum programming, verification, circuit optimization, resource analysis, and proof retrieval.

1 Purpose and scope

VQ connects algorithm constructions, formal proofs, resource analysis, and reusable research knowledge in one Lean-based repository [1]. Its current focus is quantum elliptic-curve arithmetic and the elliptic-curve discrete logarithm problem (ECDLP). Its two goals are to improve algorithms and to improve the knowledge used to develop them.

Algorithm development in this setting involves choices of arithmetic, representation, workspace, measurement, and classical control. These choices affect both correctness and resource costs. For example, an arithmetic component’s meaning includes its treatment of preserved inputs and temporary storage, as well as its numerical output. VQ makes those properties explicit in specifications and connects them to the semantics of the generated quantum program. Resource functions count operations in that same program.

The algorithm library contains constructions from arithmetic components through complete point-addition and ECDLP programs. It supports proofs about reversible action, measured execution, workspace restoration, recovery probability, and logical resources. Checked secp256k1 results include two complete runtime point-adders and ECDLP constructions with one-run recovery probability greater than 99/100 under the hypotheses in Section 5. Supporting results bound lookup costs, classical decoding, and Euclidean schedules.

The knowledge base preserves reusable contributions to that development: lemmas, proof methods, applicability conditions, procedures, and research questions. An entry can begin with an idea or a

paper to investigate and acquire definitions and evidence as work proceeds. Selected modules and entries support local tasks, while retained alternatives preserve choices of implementation and proof method.

A longer-term goal is to support collections of agents exploring algorithm variants and revising shared knowledge. Such a search would compare algorithm candidates by their proved behavior and resource costs, and knowledge revisions by evidence of their use in finding and applying results. VQ currently provides the versioned sources, component interfaces, KB-use procedure, and experiment records for these investigations.

2 System organization

VQ organizes formal constructions, reusable knowledge, and investigation records into an algorithm library, a knowledge base (KB), and experiments (Table 1). The architecture uses Lean source, Markdown documents, and supporting evidence files in one Git repository.

Collection	Connections
Library	A construction’s specification and proof identify its admitted inputs and behavior. Resource declarations describe its generated program.
Knowledge base	An entry connects a result or question to its sources, applicability, alternatives, and supporting investigations.
Experiments	A record identifies the inputs, attempts, and evidence behind a construction, comparison, or knowledge revision.

Table 1: Objects and connections in VQ’s three collections.

A library declaration has a precise statement and proof dependencies. An associated KB entry can explain the statement’s use in a caller, compare another construction, or identify a question suggested by the result. One entry can cite several declarations, and one declaration can appear in several topic guides. This supports retrieval by operation, representation, workspace requirement, or proof method across the source hierarchy.

An experiment record identifies a question, the selected source and KB revisions, the attempted changes, and the resulting evidence. A comparison records each candidate’s assumptions, resource model, and proof results. The same investigation can produce a library theorem and revise an entry’s applicability conditions. Several experiments can contribute evidence to one entry.

The active library resides under `library/VQ/`, with subject directories for quantum foundations, arithmetic, Euclidean algorithms, curve operations, and ECDLP. Worked examples and retained candidates have separate directories within `library/`. The KB stores entries under `kb/entries/` and overlapping topic guides under `kb/topics/`. Experiments reside under `experiments/`, grouped by subject and investigation. Git identifies versions of all three collections, and the root Lake workspace pins Lean and its dependencies. An individual module can be imported and checked with its dependencies.

3 Algorithm library and formal foundations

3.1 Constructions and alternatives

The library follows the mathematical decomposition of the elliptic-curve applications. Its quantum foundations include Fourier transforms, phase estimation, and probability. Reversible arithmetic supplies adders, comparisons, multiplication, squaring, constant arithmetic, and lookup. Euclidean con-

structions provide inversion, coefficient reconstruction, recorded histories, and measured workspace erasure. Curve modules combine field arithmetic into point operations. ECDLP modules combine scalar computations, Fourier sampling, and classical recovery with whole-program correctness and resource results.

A reusable component has a mathematical operation, an encoding of its operands, and a program acting on that encoding. Its theorems identify admitted input values, preserved registers, clean or borrowed workspace, and allocation. A caller proves these hypotheses when composing the component with other operations. Alternative constructions can compute the same value with different decompositions, arithmetic directions, workspace requirements, or uses of measurement. Section 3.5 develops one such comparison.

Constructions and proofs share subject directories. Worked examples retain small compositions and comparisons of their interfaces. Retained research candidates preserve alternative definitions and source identities, including partial developments. The formal results associated with each candidate refer to the declarations that establish them.

3.2 Program representations and semantics

VQ represents quantum algorithms as Lean data and circuit-generating functions. The circuit language and measured-program language share the gate representation. A generated circuit declares its quantum width and a finite gate list. The gate language includes Hadamard, Pauli, phase, controlled-X, and controlled-controlled-Z (CCZ) operations. Well-formedness predicates check operand bounds, required wire distinctness, and the phase level used by the semantics.

Exact amplitudes at phase level $\ell \geq 3$ belong to the ring

$$D_\ell = (\mathbb{Z}[X]/(X^{2^{\ell-1}} + 1))[1/2].$$

The class of X represents a primitive 2^ℓ th root of unity. The ring contains $1/\sqrt{2}$ and the dyadic phases available at that level. VQ connects this exact algebra to complex-valued quantum semantics and probabilities. Symbolic identities over the circuit generators and state semantics support proofs at cryptographic register widths.

Reversible arithmetic has a separate representation using X, CNOT, and Toffoli gates acting on natural-number basis indices. Register layouts specify which bits encode each field. The reversible compiler replaces a Toffoli with Hadamard–CCZ–Hadamard, and a semantic correspondence theorem relates the compiled quantum action on a basis state to the reversible action. This correspondence lets arithmetic proofs use bit-level and integer reasoning while retaining their meaning in quantum semantics.

Measured programs add measurement, reset, mutable classical bits, immutable classical inputs, and branches controlled by classical bits. Their semantics produces a finite list of execution branches. Each branch contains an outcome record, classical state, and subnormalized quantum state. Its squared norm determines its probability after interpretation in the complex numbers. This representation accounts for intermediate measurement and classically controlled correction within arithmetic, as well as terminal sampling.

3.3 Specifications and composition

VQ’s register specifications describe the behavior of a component on its admitted inputs. A specification contains a width w , an input predicate $A(i)$, and an output relation $R(i, j)$ on basis indices. The relation can state the computed value, preservation of other fields, and restoration of workspace. A reversible implementation meets the specification when its basis action satisfies R for every in-width input satisfying A .

For a measured implementation, the predicate `RealisesAt` also constrains branch amplitudes. At a fixed immutable classical input, it chooses an output map f and an amplitude map α before quantifying over admitted basis inputs. For an input i and terminal branch b , with outcome record r_b and quantum state ψ_b , it requires

$$R(i, f(i)), \quad |\psi_b\rangle = \alpha(r_b)|f(i)\rangle, \quad \sum_b \|\psi_b\|^2 = 1.$$

The same amplitude map applies to every admitted input. The library’s linearity theorem then establishes the corresponding action on two-term superpositions. This input-independent branch factor expresses the coherence needed when measured arithmetic forms part of a larger quantum computation.

The predicate `ImplementsRetainedU` describes measured components that preserve a specified prefix of mutable classical storage. Its parameters identify the quantum domain, reference action, retained bits, and amplitude contribution of newly appended outcomes. Earlier classical results can therefore remain part of a component interface together with its quantum action and the amplitudes of new measurement outcomes.

Composition theorems combine component actions when the first output meets the next input condition and the intermediate register lies within the declared width. Placement theorems transport local constructions into larger registers under range, injectivity, and disjointness hypotheses. These interfaces make input representation, workspace, and preserved state part of the proof that components fit together.

3.4 Logical-resource accounting

VQ computes logical-resource reports from the program syntax to which a correctness theorem refers. Circuit reports include declared width, touched wires, gate counts, and dependency depth. Measured-program reports also include measurements, resets, mutable classical storage, and immutable input width. A compiled reversible Toffoli contributes one primitive CCZ to the Toffoli-class count.

Static additive costs form intervals. Sequential composition adds costs, and a classical branch takes the minimum of its alternatives’ lower bounds and the maximum of their upper bounds. Equal endpoints give an exact count. The execution-cost semantics also attaches an accumulated cost to each terminal branch.

For a program P , classical input x , and basis input i , let p_b be the probability of terminal branch b and C_b its accumulated cost for a selected additive metric. The expected cost is

$$\mathbb{E}[C \mid x, i] = \sum_b p_b C_b.$$

VQ proves that this expectation is bounded by the static upper endpoint when P is well formed and i lies within its width. A singleton static interval gives equality. A separate definition averages over a specified finite distribution of inputs.

These reports state costs in VQ’s logical program model. Quantum width counts the declared register extent, and dependency depth follows operand dependencies. A resource theorem identifies its program, metric, and hypotheses, so a comparison can express a tradeoff between gate count, allocation, and measurement under the same semantics.

3.5 Squaring components and their comparison

The compact and placed Karatsuba square accumulators implement

$$(y, x) \mapsto ((y - x^2) \bmod p, x), \quad p = 2^{256} - 2^{32} - 977,$$

for canonical operands $0 \leq x, y < p$. Both use target positions 0–255 and source positions 256–511. The compact interface has declared width 1,488 and requires positions 512–1461 to start at zero. The placed Karatsuba interface has width 1,593 and requires zero fields at 512–1477 and 1482–1592. These conditions require 950 and 1,077 clean workspace positions, respectively. Each reference action changes the target, preserves the source and other fields, and restores its required workspace.

Subtraction uses the inverse of the compact forward construction and the forward Karatsuba construction. The comparison theorem first derives the compact input condition from the stronger Karatsuba condition, then proves equality of their reversible reference actions on that domain. The equality uses the full basis index, so it includes preservation of fields beyond the numerical result. Separate measured implementation theorems connect each operation sequence to its reference action on admitted basis inputs within the declared width. They assume phase level at least three and a scratch classical bit within the classical allocation and beyond the retained prefix.

The compact cost theorem gives 106,250 Toffoli-class operations, and the placed Karatsuba theorem gives 66,642. The comparison derives a difference of 39,608 operations. The lower count accompanies the larger workspace and allocation requirements. A measured caller uses the separate implementation and placement results to establish those requirements in its own register.

The associated KB entry records the direction of each operation, the stronger common input condition, and the declarations used in the comparison. It also records an interface detail relevant to later proofs: the two cost theorems take their arguments in different orders. The library supplies the proved equality and count difference, while the entry explains the conditions under which that comparison applies.

4 Knowledge representation and evolution

4.1 Entries, claims, and evidence

The KB represents reusable research knowledge as linked entries. An entry can contain a lemma and its formal source, a proof method, a construction, an applicability condition, a reference, or an unresolved question. Its directory supplies an identity. A short Markdown document can begin with a single idea or a paper and a question. Supporting files and additional structure follow the claims the entry makes and the information needed to find or apply them.

Evidence attaches to individual claims. A theorem link identifies a proposition and its hypotheses. A resource result identifies a program and cost model. A counterexample identifies the claim and input it refutes. An entry can therefore contain an open question together with proved supporting lemmas and evidence about attempted approaches. Relationships between entries record prerequisites, alternatives, and questions shared across subjects.

Topic guides collect these relationships around a mathematical operation or development question. The squaring guide connects integer identities, modular constructions, measured composition, and point-addition placement. The coefficient-reconstruction guide connects Euclidean recurrences, recorded histories, and normalization questions. Entries can appear in both a subject guide and a guide about a proof or implementation constraint.

4.2 Revision and reusable procedures

A KB revision can correct a claim, refine its applicability, add a construction or example, or change a procedure. Experiment records retain the observation and source revision supporting the change. Entries can split or combine as their subject develops, with earlier identities linked through retained records and Git history. A failed approach can restrict a later proposal, while a successful application can supply a reusable example or expose an unstated premise.

The squaring comparison in Section 3.5 supplied a procedure refinement: the component-reuse procedure now includes comparing input predicates before equating actions and inspecting the selected direction and cost theorem arguments. The component entries retain the construction-specific layouts and declarations. The experiment records why the shared procedure changed.

A skill is a procedure an agent can follow. VQ stores its KB-use skill as a KB entry with a `SKILL.md` document. It specifies finding relevant knowledge, checking its sources, applying a result or investigating a question, and retaining contributions. Subject-specific procedures reside in their corresponding entries, including component reuse and measured composition. The shared skill can change when recorded use reveals missing or misleading guidance.

An agent investigation records the source and knowledge revisions it used, the entries applied or rejected, and the resulting proofs or observations. Those records support review of a proposed knowledge change. Versioned skills and entries also make the research procedure available for investigation alongside the algorithm: an experiment can examine whether a description or example enables a consumer to find and apply a result.

5 Checked results

VQ contains checked functional, probability, and logical-resource results for secp256k1 point addition and ECDLP, and bounds for arithmetic and classical recovery components. Each result refers to a named program or arithmetic model. Toffoli-class costs count primitive CCZ operations in the compiled program. Phase gates retain their exact semantics at the stated phase level.

5.1 Complete runtime point addition

The wide and compact runtime point-adders implement secp256k1 group addition, including infinity and exceptional cases. Both assume a valid classical addend, a valid encoded affine accumulator, zeroed quantum workspace, a basis input within the declared width, and phase level $\ell \geq 3$. Their complete theorems establish well-formedness, the intended output with probability one, restored quantum workspace, and the bounds in Table 2. The wide implementation’s coherence theorem gives a common branch-amplitude factor for supported target states at a fixed valid addend.

Resource	Wide	Compact
Quantum width w	1,601	1,488
Mutable classical bits	13,342	13,342
Immutable classical input bits	512	512
Toffoli-class upper bound $C_{\max}$	3,070,200	4,561,806
Expected Toffoli-class upper bound	3,070,200	4,561,806
Measurement upper bound	1,959,699	2,413,324
Upper bound on $w \mathbb{E}[C]$	4,915,390,200	6,787,967,328

Table 2: Checked logical resources for the two runtime point-adders. The expectation is over measurement outcomes at a fixed admitted input.

The wide implementation exchanges 113 additional quantum positions for lower Toffoli-class and measurement bounds. Both implementations restore the quantum workspace beyond the 512-bit point encoding.

5.2 ECDLP recovery and resources

Let G be the secp256k1 generator of order q , and let the valid public point satisfy $Q = dG$ with $0 < d < q$. At phase level $\ell \geq 257$ and from all-zero initialization, the recorded-history and packed-affine constructions have the checked recovery and resource results in Table 3. Both incorporate Q during program construction. The packed-affine entry uses a 26-bit initial lookup prefix, the choice minimizing the uniform Toffoli-class bound within its family.

Property	Recorded history	Packed affine
Quantum width	1,488	833
Mutable classical bits	1,025	768
Immutable classical input bits	0	0
Toffoli-class upper bound	257,008,864	16,923,769,370
Expected Toffoli-class upper bound	257,008,864	16,923,769,370
One-run recovery probability	$> 99/100$	$> 99/100$

Table 3: Checked ECDLP results under the hypotheses in the text. The packed-affine count includes the 26-bit initial lookup. Its CNOT and public-table storage bounds appear in Table 4.

The recorded-history complete theorem connects the generated program to its bounded classical decoder. Its quantum costs cover initialization, both scalar computations, point translation, and quantum cleanup. It performs exactly 105,732,512 measurements.

The packed-affine family uses an initial table lookup and five-bit scalar windows. Under the same public-point, phase-level, and initialization conditions, its execution and recovery theorems establish restored quantum workspace and recovery with the probability in Table 3. These results have a recorded complete check at source revision `c46c2892`.

For the packed-affine program, let $c \in \{0, \dots, 25\}$ count the five-bit windows absorbed into the initial lookup, whose prefix has $k = 5c + 1$ bits. The resource theorems give an exact public-point-dependent count $C(c, Q)$ shared by every execution path, and hence by the expected cost. They prove the uniform bound

$$C(c, Q) \leq B(c) = 171,630,174 + (102 - c) 172,010,622 + 2^{5c+1} - 2$$

and $B(5) \leq B(c)$ throughout this family. Table 4 gives selected instances and the initial-lookup bounds. The lookup stores each public point in 512 bits. Its table storage is $512 \cdot 2^k$ bits, and the initial lookup uses at most $514 \cdot 2^k - 2$ CNOTs.

Prefix bits k	Whole-program $B(c)$	Initial-lookup CNOT bound	Public table
11	17,372,694,420	1,052,670	128 KiB
16	17,200,747,286	33,685,502	4 MiB
21	17,030,768,280	1,077,936,126	128 MiB
26	16,923,769,370	34,493,956,094	4 GiB
31	18,832,133,532	1,103,806,595,070	128 GiB

Table 4: Checked lookup tradeoffs for the 833-qubit measured ECDLP family. Public-table storage counts packed point data.

The bounded decoder has a separate classical-operation result: 514 scalar-multiplication calls, at most 66,049 candidate comparisons, and at most one modular inversion for each sampled pair.

5.3 Euclidean termination and schedule bounds

For an odd prime $p < 2^n$ and $0 < x < p$, the recorded binary Euclidean trace reaches $(1, 0)$ within $2n - 1$ steps. The proof uses descent of the sum of the two register bit lengths. At $n = 256$, the bound is 511 steps.

The quotient-based Euclidean model has a different step accounting. For ordinary Euclidean quotients $q_1, \dots, q_D$ starting from $0 < b \leq a$, define

$$W(a, b) = \sum_{j=1}^D \text{bitLength}(q_j), \quad E(a, b) = \sum_{j=1}^D (\text{bitLength}(q_j) - 1).$$

Here $\text{bitLength}(0) = 0$ and $\text{bitLength}(u) = \lfloor \log_2 u \rfloor + 1$ for $u > 0$. The general bounds give $W = E + D$, $E \leq \text{bitLength}(a)$, $D \leq 2 \text{bitLength}(a)$, and $W \leq 3 \text{bitLength}(a)$. For a prime $p < 2^{256}$ and $0 < a < p$, preprocessing uses $x = \min(a, p - a)$. The preprocessed schedule theorem proves $W(p, x) \leq 405$ and the associated round bound $4W(p, x) \leq 1620$. The 511-step and 1,620-round bounds refer to their respective transition systems.

6 Related work

VQ combines functions that other systems provide through quantum programming languages, proof environments, circuit optimizers, resource analyzers, and algorithm libraries. Table 5 compares the objects and methods of those systems. The comparison uses the versions described in the cited publications. A proof about a program family, an equivalence check between two circuits, and a resource estimate from a component decomposition establish different properties.

6.1 Algorithm libraries and component development

Qualtran [13] provides typed algorithm components with protocols for decomposition, call graphs, simulation, and resource analysis. These protocols can be supplied independently. A call graph can specify subroutine multiplicities before the full dataflow decomposition exists, and recursive counting caches repeated subroutine costs. This supports analysis at algorithm scale while components acquire further detail. Its library includes arithmetic and elliptic-curve discrete-logarithm constructions, giving a direct comparison with VQ’s application domain.

Qualtran and VQ both retain reusable components at different stages of development. Qualtran’s decomposition and analysis protocols organize the information available about a component. VQ associates formal claims with Lean declarations and connects those declarations to applicability conditions, alternatives, and research records in the KB. In a VQ substitution, the comparison theorem states the common input domain and reference action, as in Section 3.5. Resource comparisons refer to the resulting programs and their allocation requirements.

6.2 Quantum languages and proof environments

SQIR and its verified optimizer VOQC [3] embed indexed circuits in Coq and use symbolic matrix reasoning to prove properties of those circuits. The optimizer’s proofs quantify over circuits satisfying each transformation’s conditions, and extraction produces an OCaml optimizer. A verified pass therefore supplies a reusable transformation together with its semantic preservation result.

System	Representation	Method	Result or analysis
Qualtran [13]	Typed algorithm components, decompositions, and call graphs	Composition, symbolic counting, and simulation protocols	Reusable algorithm library, logical counts, and physical-resource estimates under specified models
SQIR/VOQC [3, 4]	Indexed quantum circuits embedded in Coq	Symbolic matrix reasoning and proved transformations	Semantics-preserving optimization. SQIR also supports a certified factoring implementation
CoqQ [5]	Quantum programs and operator assertions in Coq	Program logic proved sound against denotational semantics	Compositional functional proofs with local and parallel reasoning
Qbricks [6]	Circuit-generating programs and path-sum specifications	Deductive verification through Why3	Parameterized circuit correctness and resource properties
Qafny [7]	Typed quantum programs and array-based assertions	Translation to Dafny using a separation-logic proof system	Automated functional verification of specified quantum programs
QWIRE [2]	Linear circuit language embedded in a classical host	Operational semantics and density-matrix denotation	Type safety, normalization, and semantic preservation of circuit reduction
AutoQ 2.0 [8]	Sets of pure states represented by tree automata	Automata operations and SMT-assisted entailment	Partial correctness of programs with measurement, branching, and annotated loops
CertiQ [9]	Compiler passes over circuit representations	Circuit calculus and SMT verification conditions	Equivalence-preserving implementations of Qiskit compiler passes
PyZX [10]	ZX-diagrams representing linear maps	Graph rewriting and circuit extraction	Circuit optimization and equality validation
QuRA [11]	Circuit-generating programs with indexed types	Metric-parameterized type-and-effect analysis	Symbolic bounds on width, gate counts, and depth
Qet [12]	Mixed classical-quantum programs with cost annotations	Expectation transformers and constraint solving	Upper bounds on expected execution cost, including measured loops
LeanDojo [15]	Lean proof states, tactics, and available premises	Program analysis and retrieval-augmented proof search	Proof-environment interaction, premise selection, and tactic generation
VQ	Lean circuit generators, measured programs, and linked KB entries	Semantic proofs, component composition, and program-based cost accounting	Quantum EC constructions with functional, recovery, and resource theorems, plus versioned research knowledge

Table 5: Representations, methods, and results in systems related to VQ.

The SQIR-based factoring development [4] extends that comparison to a complete algorithm. Peng et al. verify arithmetic oracles, quantum sampling, and classical recovery. Their reversible intermediate representation supports arithmetic proofs on basis states and a correspondence with quantum semantics. VQ uses the same division of proof responsibilities through its reversible language and compiler correspondence. Both developments connect arithmetic implementations to a probabilistic application result under explicit input and recovery conditions. For arithmetic containing intermediate measurements, VQ’s component interfaces specify retained classical data and input-independent outcome amplitudes.

CoqQ [5] provides operator assertions, including Dirac expressions, and a program logic proved sound against denotational semantics. Its formalization builds on Mathematical Components and its analysis library. Local and parallel reasoning support proofs about commands acting on parts of a state. VQ supplies register specifications and component theorems about encoded arithmetic, workspace restoration, and branch amplitudes.

Qbricks [6] verifies circuit-generating programs through parameterized path-sum specifications and Why3 verification conditions. Its examples include phase estimation, Grover search, and the quantum order-finding part of Shor’s algorithm. The order-finding development includes arithmetic oracles and a polynomial circuit-size result. A proof about a circuit generator applies across its admitted parameters. VQ likewise proves properties of generating definitions, with Lean declarations stating the width and arithmetic hypotheses. Its complete EC results instantiate cryptographic parameters and give numerical bounds for the generated logical programs.

Qafny [7] translates typed quantum programs into classical array operations for verification with Dafny. Types guide the translation, and the paper proves soundness and completeness of its quantum proof system. Examples include quantum walks, Grover search, and Shor’s algorithm. Qafny and Qbricks use classical deductive-verification tools to discharge quantum proof obligations through their respective encodings. VQ expresses the program semantics and arithmetic proofs in Lean, where reusable lemmas and tactics construct proofs of the imported definitions.

QWIRE [2] separates a linear quantum circuit language from its classical host and permits measurement results to become host-language values. Its type-safety and normalization results assume the corresponding host properties, while density matrices provide the circuit denotation. The host-language boundary determines how generation and classical computation enter the formal account. VQ uses Lean functions for generation and explicit measured-program syntax for measurement, classical storage, and conditional execution.

AutoQ 2.0 [8] represents sets of pure states with tree automata. Its procedure requires preconditions, postconditions, and annotated loop invariants, and checks entailment using automata operations and SMT. It compares pure states up to a nonzero complex factor, accommodating the normalization associated with measurement. VQ’s recovery and expected-cost results require the probability assigned to each branch. Its semantics retains amplitudes and interprets their squared norms as branch weights. State-relation verification and probability-weighted claims thus require different information about measured execution.

6.3 Circuit transformation systems

CertiQ [9] verifies compiler passes using a circuit calculus and SMT verification conditions. Its coverage includes the data structures, transformation functions, and conversions between representations used by the passes. VOQC proves transformations over SQIR syntax in Coq. In both systems, a pass proof establishes a relation between input and output circuits for the pass’s admitted inputs.

PyZX [10] represents linear maps as ZX-diagrams and combines graph rewriting with circuit extraction. Diagram structure permits transformations through representations that differ from a gate sequence, with equality validation supporting optimization. These systems organize reuse around

transformations and equivalence checks. VQ’s component alternatives also include choices of arithmetic decomposition, allocation, and measured cleanup. Their comparison states a relation between actions on admitted register states and accounts for the resources of each resulting program. Placement and representation theorems establish the conditions for using a selected component within its caller.

6.4 Resource analysis

QuRA’s [11] Proto-Quipper-RA type system separates global circuit resources from quantities attached to individual wires. Effects describe the global resources, and indexed wire types describe the wire-local quantities. The soundness theorem requires inequalities connecting resource-index interpretations to the chosen metric. The published implementation supports both forms of analysis, permitting symbolic bounds on width, gate counts, and depth.

VQ defines resource functions in Lean and proves their relation to execution costs. Its width counts the declared register extent. QuRA’s circuit-width metric counts the maximum number of active wires. A numerical comparison therefore requires agreement on allocation and reuse as well as the gate metric. Qualtran’s resource counts derive from component decompositions or call graphs, with physical-resource estimates depending on further cost models. These numbers describe the objects and assumptions supplied to each analysis.

Qet [12] analyzes expected execution costs of mixed classical–quantum programs. It represents density-matrix and probability expressions symbolically, generates upper-invariant constraints for loops, and reduces them to polynomial conditions for which SMT solving seeks certificates. An inferred expectation can depend on the input state. Its repeat-until-success examples include measurement-dependent execution lengths.

VQ’s expected-cost semantics sums costs over a finite list of execution branches. Structural intervals provide static bounds, and a theorem bounds the expectation by the static upper endpoint. The packed-affine ECDLP resource result additionally proves a common Toffoli-class count for every path, giving an exact expectation for a fixed public point. Qet’s loop analysis and VQ’s finite-branch cost proofs address different program classes and forms of cost reasoning.

6.5 Reusable proofs and research knowledge

The Lean mathematical library, `mathlib` [14], provides definitions, theorems, and tactics organized through Lean’s module and dependency structure. VQ develops its formal library in that setting. A formal declaration identifies a proposition and its proof dependencies. VQ’s linked KB entries add application conditions, comparisons, research questions, and procedures, with evidence attached to individual claims.

LeanDojo [15] extracts Lean proof data, including premises used by tactics, and provides programmatic interaction with the proof environment. Its ReProver system retrieves accessible premises to support language-model proof search. This gives a comparison at the level of finding and applying formal knowledge. VQ’s KB-use skill describes a broader research procedure that includes source assessment, component selection, and contributions to the KB after an investigation. The objects retained for later use include both formal declarations and records of the reasoning and evidence behind a research choice.

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