

VQ: Quantum Algorithm Development and Verification with an Evolving Knowledge Base

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Abstract

Developing quantum elliptic-curve algorithms requires repeated changes to arithmetic, allocation, cleanup, and recovery proofs. VQ combines Lean constructions and proofs with an evolving knowledge base and retained experiments to support local algorithm development and verification. Its semantics connects reversible basis action, exact quantum amplitudes, measured execution, and logical-resource functions. The current checked applications include complete secp256k1 point addition and a 1,488-qubit ECDLP program with one-run recovery probability above 99/100 and a Toffoli-class upper bound of 257,008,864. A separate 833-qubit result retains its earlier complete check. The knowledge base records applicability conditions, procedures, open questions, and skills beside links to source and evidence. Squaring consumers exercise allocation and measured composition. A coefficient-reconstruction trial derives a quotient-recovery theorem and a conditional implementation budget. These trials support specific reuse and knowledge revisions. Comparative agent-effort measurements remain open. The report identifies current build failures, unfinished checks, and the further proofs required for algorithm substitutions and physical-resource claims.

1 System scope and evidence

VQ combines Lean constructions and proofs with an evolving knowledge base and retained experiments to support local algorithm development and verification. The working objects are generated circuits, measured programs, theorem statements, source notes, and records of attempts. A task can start with a proposed arithmetic identity, a component to replace, or a paper to investigate. Its result can change a construction, establish a comparison, correct a procedure, or narrow an unresolved question [2, 3].

This report describes repository revision 12640cb5 on the `reorg` branch [1]. Active Lean source, dependency pins, and the runner configuration have the same bytes as source-migration revision 7a494241. The recorded checks at that revision used Lean 4.34.0-rc2. The subsequent review corrected documentation and evidence retention. The present report draws its proof-checking results from those retained runs [5, 6].

A declaration specifies the proposition and its hypotheses. A completed Lean check establishes acceptance of that declaration and its imports under the recorded configuration. An experiment records which knowledge a task used, the attempts it made, and the result it retained. These forms support different conclusions: a theorem about one component, for example, leaves the caller’s placement and input invariants to the caller [4].

2 Exact semantics and component proofs

2.1 Circuits and finite measured programs

The circuit language contains Hadamard, Pauli, phase, controlled-X, and controlled-controlled-Z gates. A circuit declares its width and gate list. At phase level $\ell \geq 3$, its amplitudes belong to

$$D_\ell = (\mathbb{Z}[X]/(X^{2^{\ell-1}} + 1))[1/2].$$

The class of X represents a primitive 2^ℓ th root of unity. The ring contains $1/\sqrt{2}$, and selected higher dyadic rotations use roots available at that level. Well-formedness checks the level, operand bounds, and required wire distinctness [7].

Reversible arithmetic uses a separate syntax with X, CNOT, and Toffoli gates acting on natural-number basis indices. A well-formed circuit preserves the width bound and has an inverse obtained by reversing its gate list. The compiler replaces each Toffoli by Hadamard–CCZ–Hadamard. Its semantic theorem identifies the compiled quantum action on a basis state with the reversible action [8].

A hybrid program declares quantum width, mutable classical storage, immutable classical input width, and a finite list of operations. Operations include gates, measurement, reset, classical stores and inversions, and branches on classical bits. Execution produces a list of branches, each carrying an outcome record, mutable classical state, immutable input, and a subnormalized quantum state. Finite circuit generators express repetitions by constructing operation lists [7].

Symbolic proofs follow the arithmetic generator, register invariants, and finite-sum identities. This supports cryptographic-width reasoning while direct state evaluation remains useful for small instances. A circuit with w wires has 2^w basis indices, and a sequence of measurements can generate an exponential number of branches. The complete algorithm proofs use identities for these states and distributions [14, 15].

2.2 Register relations and coherent measured action

A register specification consists of a width w , an input predicate $A(i)$, and an output relation $R(i, j)$. The relation can express the result value, preserved fields, and restored workspace in one complete basis index. For a fixed immutable classical input x , the predicate **RealisesAt** supplies an output map f and a record-amplitude map α before quantifying over admitted basis inputs. Each terminal branch b satisfies

$$R(i, f(i)), \quad |\psi_b\rangle = \alpha(r_b)|f(i)\rangle, \quad \sum_b \|\psi_b\|^2 = 1,$$

where $i < 2^w$, $A(i)$ holds, and r_b is the branch’s outcome record [9]. The quantifier order fixes the same amplitude map for all admitted i . Linearity then transfers the action to supported superpositions with the corresponding record-dependent factor.

Measured arithmetic also uses **ImplementsRetainedU**. Fix an incoming classical register c_{in} and let $\bar{c}$ be its low r retained bits. Let c_b denote a branch’s final mutable classical state. For a branch that adds m_b outcomes to the incoming record, this predicate states

$$\text{low}_r(c_b) = \bar{c}, \quad |\psi_b\rangle = a^{m_b}|f_{\bar{c}}(i)\rangle. \quad (1)$$

The domain, width, immutable input, retained prefix, output map, and amplitude base a are parameters. Measured-uncomputation components commonly use $a = 1/\sqrt{2}$. Normalization, well-formedness, and bounds needed by a complete program have their own theorems [10].

Composition requires the first output to satisfy the second component’s input predicate and remain in range. For the point-specialized interface **append_point**, the inputs are endpoints $i \mapsto j$ and $j \mapsto k$, with a common width and retained-state interface, plus $j < 2^w$. Concatenation then establishes $i \mapsto k$

in the same measured predicate. Placement theorems transport local actions into larger registers under explicit range, injectivity, and disjointness conditions [10, 11].

3 Logical-resource claims

Resource functions traverse the same syntax named by semantic theorems. Circuit reports include declared width, touched wires, gate classes, and dependency depth. Program reports add measurements, resets, mutable storage, immutable input, and classical control. Sequential additive costs add, and classical alternatives produce lower and upper endpoints. The upper endpoint therefore includes the more expensive syntactic arm even when an application always chooses the other arm [12].

An execution-cost semantics pairs each terminal branch with its accumulated additive cost. For a fixed classical input x and basis input i , write p_b for the branch’s real weight and C_b for its chosen cost. The expected cost is

$$\mathbb{E}[C \mid x, i] = \sum_b p_b C_b.$$

The theorem `inputExpectation_le_static` bounds this sum by the static upper endpoint for a well-formed program and an in-width basis input. If the static range is the single value t , `inputExpectation_eq_of_range_point` gives expected cost t . A further definition averages over a specified finite input law [13].

In this report, a Toffoli-class count is the count of primitive CCZ operations in the compiled program. The reversible compiler contributes one CCZ per Toffoli. Measurement-assisted arithmetic has its own generated operations, semantic proofs, and counts. Quantum width denotes the declared register extent. Dependency depth denotes the logical schedule computed from operand dependencies. Phase synthesis, routing, error correction, device timing, and classical implementation costs require additional models and theorems [8, 12, 16].

An optimization comparison must identify the domain, representation, preserved registers, allocation, and resource function for both candidates. A reduced component count can require more workspace or change the surrounding cleanup. The complete caller supplies the resulting algorithm-level bound. Section 7 gives a case in which a checked arithmetic subtotal leaves both implementation costs and caller eligibility open.

4 Complete secp256k1 algorithms

The current recorded checks establish complete runtime point addition and a recorded-history ECDLP program. A separate packed-affine ECDLP construction retains an earlier complete check. The bounds in Table 1 count logical Toffoli-class operations in one invocation [14, 15, 16, 5].

Construction	Qubits	Toffoli upper bound	Complete-check record
Runtime point addition, wide	1,601	3,070,200	Passed after source migration
Recorded-history ECDLP	1,488	257,008,864	Passed after source migration
Packed-affine ECDLP, 26-bit initial lookup	833	16,923,769,370	Passed at <code>c46c2892</code> . Migrated check unfinished

Table 1: Logical bounds and their recorded verification status.

The runtime point-adder takes a classical addend encoded in 512 immutable bits and acts on an affine quantum accumulator. Its valid-point domain includes infinity and the exceptional group-law

cases. The wide complete theorem assumes a valid addend, a ready accumulator and workspace, an in-width basis input, and phase level at least three. It establishes the intended output with probability one, zeroed quantum workspace, and the resource bounds. The associated superposition theorem preserves immutable input and proves a common branch amplitude for every supported target superposition at that fixed addend. The program uses 13,342 mutable classical bits and at most 1,959,699 measurements [14].

For ECDLP, let G be the secp256k1 generator, q its order, and $Q = dG$ a valid public point with $0 < d < q$. At phase level at least 257, the recorded-history program starts from zero, performs both scalar computations and semiclassical Fourier sampling, and passes the outcomes to a bounded classical decoder. The theorem `measured_complete_result` proves one-run recovery probability greater than 99/100, the width in Table 1, 1,025 mutable classical bits, and exactly 105,732,512 measurements. The expected Toffoli-class count is bounded by the same stated upper bound [15].

The packed-affine result uses 833 qubits, 768 mutable classical bits, a public point lookup, five-bit scalar windows, and semiclassical Fourier sampling. The 26-bit initial lookup minimizes the proved uniform Toffoli bound within the studied prefix family. Its initial lookup alone has a CNOT upper bound of 34,493,956,094 and a packed table size of 4 GiB. Under the same valid-key, nonzero-secret, zero-initialization, and phase-level assumptions, the recorded complete result proves recovery probability above 99/100. That result is pinned to `c46c28927244d3cb1ee664677a0f61145ca45d54`; the subsequent migration checks reached their resource limits [16, 5].

Both ECDLP programs specialize their generated syntax to the public point and declare zero immutable input bits. The resource bounds leave public-table generation and storage, classical decoding costs, full-program CNOT totals and depth, and physical execution costs for separate accounting. The exact Fourier phase model also leaves synthesis error and its effect on terminal recovery to a further analysis [15, 16].

5 Library, knowledge base, and experiments

The repository keeps active constructions in one Lake workspace. Source paths group quantum foundations, arithmetic, Euclidean algorithms, curves, and ECDLP. Worked consumers demonstrate local reuse. Earlier variants retain source identities and their original verification records [2].

Location	Content and use
<code>library/VQ/</code>	Active constructions, semantic definitions, and proofs grouped by subject
<code>library/examples/</code>	Small consumers and the separate Challenge/Solution submission example
<code>library/candidates/</code>	Retained variants with source provenance and preparation requirements
<code>kb/entries/</code> and <code>kb/topics/</code>	Reusable knowledge and overlapping subject guides
<code>experiments/</code>	Questions, inputs, attempts, checks, results, and supporting evidence

Table 2: Repository collections used by development tasks.

A knowledge-base (KB) entry can begin with one question or a reference to research. As it develops, it records applicability, source declarations, evidence, limits, and examples where those help a consumer. An entry can contain a lemma, tactic, procedure, skill, construction, or open question, and can have several of these roles. Topic guides provide overlapping routes to entries and source. The directory supplies an entry’s identity, and Git retains its revisions [3].

The KB-use skill is an entry at `kb/entries/use-kb/SKILL.md`. It directs a task to locate relevant knowledge, inspect the sources supporting each claim, apply or investigate it, and retain the outcome. It also tells the consumer to distinguish value identities, reversible action, measured execution, cleanup, probability, and resources. The procedure can change when recorded use exposes missing guidance [3].

An experiment records the question, selected source and KB revisions, relevant uncommitted changes, commands, full output, exit status, and remaining obligations. It identifies entries used, adapted, or rejected and the reason. The result can revise an existing entry without creating a new general lemma. Failed attempts remain useful when they explain a correction or delimit the next question [4].

Module imports follow current source paths, while declaration namespaces keep their written names, including older `VQMathlib` and `VQBridge` namespaces. A consumer therefore finds a declaration in its defining module and imports that module. Building a selected module checks its dependency closure. The root `VQ` target and the default `Challenge` and `Solution` targets have their own recorded status [2, 5].

6 Squaring reuse and measured composition

The first squaring comparison relates compact square accumulation to a placed Karatsuba construction. Their forward directions differ: compact inverse execution and forward Karatsuba execution both subtract a square. Write $\text{act}(C, i)$ for the reversible basis action of circuit C on input i . Under the stronger Karatsuba readiness condition, the consumer proves

$$\text{act}(\text{Karatsuba}, i) = \text{act}(\text{Compact}^{-1}, i).$$

The compact inverse sequence uses exactly 39,608 more Toffoli-class operations than the forward Karatsuba sequence. The proof derives the compact clean-workspace condition from the larger caller condition, then rewrites with the two reference-action theorems. The comparison concerns the supplied interfaces and their operation counts [18].

Allocation determines which interface a caller can use. The compact measured construction has declared width 1,488 and requires 950 clean positions. The local Karatsuba construction has width 1,589, and its flag-preserving placed interface has width 1,593. A supplied allocation above a caller’s limit therefore requires a new placement proof before use at that limit. The topic guide records these distinctions alongside the value and measured declarations [17].

A fresh agent received a 1,500-qubit limit, the KB-use skill, and a task to compose square subtraction with its measured inverse. It selected the compact construction, assumed canonical operands and 976 clean caller positions, and derived the narrower component condition. Let i be the initial state and j the compact inverse reference action. Inverse readiness establishes the component domain at j , cancellation proves that forward execution returns i , and an in-width theorem supplies the premise for `append_point` [19].

The consumer proves equation (1) with output i for the subtraction–addition sequence. It also proves exactly 212,500 Toffoli-class operations and 212,488 measurements. The quantum reference state and retained classical prefix return as specified, while the outcome record grows and the scratch classical bit remains outside the preserved prefix. The recorded focused check and the migrated consumer check both passed [19, 5].

The agent began without the parent conversation and could read earlier repository examples. It needed no mathematical hints, but its first proof failed and six guessed paths produced file errors. The retained second attempt passed after it used named width and bound results. A later primary agent diagnosis isolated a circuit-action versus gate-list-action conversion. The KB gained a subtraction-first example, an inverse-readiness reference, and the checked conversion. These observations establish

reuse and revision on this task. Their comparison with a task performed without the skill remains open [19].

7 Coefficient reconstruction from a research question

The next trial investigated delayed normalization in recorded Euclidean coefficient reconstruction. The existing proposal applied a batch of signed updates before reducing the target to its canonical interval. The consumer selected histories with zero swap flags and source multiplier $+1$ or -1 , extending an earlier two-step calculation. Its result supplies a quotient recovery theorem for that restricted family [20].

Proposition 1 (Restricted quotient recovery). *Let m, p, b, T, V, Y, q be integers with $m \geq 1, p > 0$,*

$$b \leq T < b + p, \quad b \leq Y < b + p, \quad b \leq V \leq b + p.$$

Set $D = 2b + p$ and suppose

$$Y + pq = 2mT - V - (m - 1)D.$$

Then $-m \leq q \leq m$. If $\lambda = q \bmod 2m$, then

$$q = \begin{cases} \lambda, & \lambda < m, \\ \lambda - 2m, & \lambda > m, \\ -m, & \lambda = m \text{ and } Y + V \geq D, \\ m, & \lambda = m \text{ and } Y + V < D. \end{cases}$$

Proof. The raw expression lies in $[b - mp, b + (m + 1)p)$. Combining these bounds with the canonical interval for Y and using integrality gives $-m \leq q \leq m$. In this interval, reduction modulo $2m$ identifies only the two endpoints $-m$ and m . If $q = -m$, the defining equation gives $Y + V = D + 2m(T - b) \geq D$. If $q = m$, it gives $Y + V = D + 2m(T - b - p) < D$. The comparison therefore distinguishes the endpoints. The Lean declarations `quotient_range`, `ambiguous_residue`, `extreme_comparison`, and `recover_eq` check these steps [20]. $\square$

With current target U and fixed source R , bit zero denotes the unreduced update $U \mapsto 2U - R$, and bit one denotes $U \mapsto 2U + R - D$. For five updates, the selected sign words are 01111 and 10000, written in execution order with the first choice as the most significant digit. They give $2m = 2^5$. The preserved source is either the canonical coefficient $R \in [b, b + p)$ or its complement $D - R$. The source interval includes its upper endpoint: $R = b$ gives $D - R = b + p$. The agent corrected its initial strict interval after checking that case. For $p \geq 33$, a further checked collision gives equal low five bits of the canonical output target Y for quotients -16 and 16 on the generic canonical domain. It establishes the need for additional information in that domain [20].

The proposed implementation performs widened subtraction, adds an offset, normalizes by six comparison/correction passes, and erases a high quotient flag using the comparison above. Specializing existing primitive count theorems gives a subtotal of 3,902 Toffolis. Five current steps with swaps absent at construction time cost 5,145. If L denotes the Toffoli cost of all omitted or changed operations, including low-bit recovery, flag preparation and erasure, and changed caller work, the proposed schedule improves that baseline exactly when

$$3902 + L < 5145, \quad \text{equivalently } L < 1243.$$

The resource exercise checks the primitive specializations and this arithmetic. The composed circuit, allocation, measured cleanup, and complete-caller comparison remain open [20].

The history restriction also constrains integration. A caller needs to establish the selected history before constructing the restricted circuit or provide a uniform implementation that shares operations across histories. If eligibility depends on quantum data, separate controlled implementations retain the gates of both candidates in the counted syntax. The frequency of eligible histories then supplies no Toffoli-count reduction. The trial records this restriction in the existing normalization entry and links the new collision to the low-bit obstruction entry [20, 3].

This fresh agent used the revised source-location instruction but continued to make path and namespace guesses. The primary agent requested scope corrections concerning a power-of-two premise and the comparison baseline. Both focused Lean files and the relevant point-adder aggregate passed. The experiment supports the arithmetic result, conditional count calculation, and KB corrections. Causal claims about the skill’s effect on retrieval or agent effort require controlled comparisons [20].

8 Verification maintenance and current limits

The migration audit checks a fixed source revision against its recorded pre-migration input, allowing the mapped import changes. It reads the maps and source from Git, so later experiments can add Lean files while the historical preservation test keeps the same inputs. Current documentation has a separate check against the Git index or a selected revision. The log and inventory writers reject replacement of retained evidence when local ignored copies are absent [5, 6].

The baseline checks exposed stale measured-resource literals in recorded-history ECDLP. The repair propagated the current component counts through translation resources, branch amplitudes, scalar loops, and branch multiplicities, and the complete aggregate passed. The circuit generators and program operation lists remained unchanged from baseline revision 2c544bb. The resulting KB entry records the coupling between measurement counts, amplitude exponents, and branch-counting statements [21].

The broad library root currently fails in three preserved modules. `Secp256k1SevenCarry` has two gate-list equalities that evaluate as false after an earlier arithmetic change. The prefix-cleanup proof in `BorrowedCoefficient` exceeds recursion limits. `BorrowedWindowedOwnership` exceeds heartbeat limits in four equivalence proofs. The KB repair question identifies the failed statements and the hypotheses of a related action theorem [22].

The packed-affine migration checks repeatedly reached a 900-second command limit. An isolated resource-component check also timed out under the recorded memory limits. The relocated `Solution` target remains unchecked as a complete target. The runtime point-adder, recorded-history ECDLP, `Challenge`, and the nine worked-consumer checks passed. Complete output for successful, failed, and timed-out commands remains in the migration record [5].

The axiom guards for complete point addition and recorded-history ECDLP report propositional extensionality, classical choice, and quotient soundness, named `propext`, `Classical.choice`, and `Quot.sound`. Their proofs rely on the pinned Lean and imported definitions. A passing dependency check establishes its own target. The failed and unfinished aggregates retain their separate status. Independent checker results for earlier submissions have their historical records and were not repeated during the reorganization [14, 15, 5].

9 Further experiments and related work

The existing agent trials cover one bounded squaring consumer and one coefficient research question, with earlier examples available to both. Assignments, sources, failed attempts, outputs, and reports are retained. The migration record identifies the absence of a complete assistant-session transcript.

These records support the reported task outcomes. They leave general retrieval quality, agent effort, and the effect of individual skill revisions for further measurement [19, 20, 5].

A proposed extension is a population of isolated agent tasks that develop variants under common input and cost assumptions. Each candidate would retain its program, proof obligations, source revision, selected KB entries, and evidence. Selection could compare proved resource bounds subject to the caller’s allocation and recovery requirements. Combining candidates would create a further proof task for the combined program. This is a prospective search procedure. The present trials establish local reuse and contribution.

KB evolution also needs experiments. A useful comparison would fix a source revision and task, vary the supplied entries or skill, and record retrieval failures, completed proofs, effort, and the resulting knowledge changes. Independent writable worktrees and build directories would preserve each candidate’s inputs. Corrections to entries could then cite the trials that support them, while unresolved questions and failed approaches remain available [2, 4].

Prior systems establish complementary methods for quantum verification. SQIR and VOQC use Coq and symbolic complex-matrix semantics to prove circuit transformations [23]. CoqQ provides a program logic with a proved relation to its denotational semantics [24]. Qbricks uses deductive verification for parameterized circuit-building programs [25]. VQ’s contribution here is the connection between exact measured arithmetic, complete elliptic-curve applications, and experiments that revise the knowledge used by later tasks. The comparisons in this report concern those recorded VQ constructions and trials.

A Source access and reproduction

The full report input is 12640cb53cd4271f1ba55e562b3dbd61654b191b. Source links in the references select that revision. Historical packed-affine evidence selects its earlier complete-result revision. The root `lean-toolchain` and `lake-manifest.json` specify the toolchain and dependencies. The repository’s development guide and installed `leanrunner` skill specify execution and resource limits [1, 5].

The following commands select the two currently completed algorithm aggregates and the squaring agent’s consumer from the repository root:

```
tools/leanrun lake --no-ansi build \  
  VQ.Curve.RecordedAffine.WideRuntimeComplete  
tools/leanrun lake --no-ansi build \  
  VQ.ECDLP.RecordedAffine.MeasuredComplete  
tools/leanrun lake env lean \  
  experiments/kb/skill-agent-1/agent/Consumer.lean
```

The coefficient trial links its arithmetic and resource checks in its record. The migration log headers identify the checked input and commands, and each log retains the complete output and final status. The source-preservation audit connects the recorded import changes to the committed migration.

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