

VQ: Exact Functional Verification and Certified Logical-Resource Analysis of Quantum Algorithms

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Abstract

VQ is a Lean 4 framework for defining quantum circuits and hybrid quantum–classical programs, proving their behavior, and deriving logical resource results from the same executable syntax. Its semantic models cover well-formed finite circuits over exact dyadic cyclotomic amplitudes, reversible basis permutations, and programs with measurement, reset, immutable classical input, mutable storage, and feed-forward. Functional specifications state the admitted inputs, output values, preserved registers, workspace cleanup, and the input independence required for measured components to act coherently on superpositions. Static analyses report declared width, touched wires, gate classes, dependency depth, measurement, reset, and classical control, while semantic branch weights support expected additive path costs for basis inputs under finite input laws. Checked applications include parameterized arithmetic, measurement-assisted addition and lookup cleanup, secp256k1 inversion and point addition, and test-local exact quantum Fourier transform (QFT), phase-estimation, and secp256k1 elliptic-curve discrete-logarithm problem (ECDLP) developments. VQ certifies ideal finite semantics and logical resources. Its production API excludes density operators, physical noise, routing transformations, fault-tolerant costs, and useful-precision phase synthesis connected to terminal algorithm error bounds.

1 VQ states semantics and resources for generated programs

VQ treats a quantum algorithm as an executable Lean value accompanied by theorems about that value. A generator may depend on widths, moduli, layouts, or classical data, and a closed artifact fixes those parameters. The application results in this report name the generated value or connect source and target values through a proved translation [16].

VQ states functional behavior, preservation, cleanup, coherence, terminal success, and resources through explicit theorem statements. In the checked applications, semantic and resource theorems name the same generated syntax. When an application theorem names both results, Lean checks their shared program expression.

Lean defines the generators, mathematical objects, and proofs [3]. The kernel checks their proof terms against pinned Lean, Mathlib, and supporting library revisions [17]. The analyzed artifact is revision `d407e8c`, whose aggregate build checks the production roots, test root, verifier, and fixed runtime point-addition target.

The repository assigns evidence status according to API reuse and theorem quantification. Every VQ result called checked in this report is a kernel-checked theorem at the pinned revision. A closed artifact fixes its parameters and may have any of the statuses in Table 1.

Evidence status	Meaning
Production	A reusable declaration imported by a production aggregation root.
Test-local	A kernel-checked declaration imported by the aggregate test root, pending review of its public API.
Example-specific	A kernel-checked result specialized to one program, modulus, width, layout, or example package.

Table 1: Evidence status used for VQ results in this report.

2 Exact semantics cover circuits, reversible arithmetic, and hybrid programs

The unitary circuit language records a quantum width and an ordered list of primitive gates. Its gate set contains Hadamard, X, Y, Z, S, inverse S, T, inverse T, exact higher dyadic phases and their inverses, CNOT, and CCZ. Well-formedness checks wire bounds, distinct operands, and whether the selected phase level contains every root of unity used by the circuit.

At semantic level $\ell \geq 3$, amplitudes lie in the dyadic cyclotomic ring

$$D_\ell = \left(\mathbb{Z}[X]/(X^{2^{\ell-1}} + 1) \right) [1/2].$$

The class of X represents a primitive 2^ℓ th root of unity, and the ring contains $1/\sqrt{2}$. Circuit execution acts on a basis-indexed amplitude function, and the library proves linearity, well-formed support, composition, and exact denotation. The model compares exact algebraic amplitudes.

Reversible circuits use X, CNOT, and Toffoli gates and denote transformations of basis indices. The library proves that every well-formed reversible circuit acts injectively within its declared width. A verified compiler maps this syntax into the unitary language, replacing Toffoli by a Hadamard–CCZ–Hadamard sequence and proving the same basis action in the exact semantics.

Hybrid programs add measurement, reset, immutable input reads, mutable classical bits, bit inversion, and classical branches. Execution produces subnormalized quantum branches carrying their measurement record and classical state. For a well-formed program acting on a width-supported unit-norm exact state, a production theorem proves that the branch weights sum to one.

Program form	Semantic object	Principal claims
Unitary circuit	Exact linear action on pure-state amplitudes	State equality, basis-input full-register and selected-wire probability, test-local semantic adjoint laws, and gate-level resources
Reversible circuit	Injective permutation of bounded basis indices	Arithmetic action, preserved fields, cleanup, and compilation to unitary gates
Hybrid program	Finite list of subnormalized branches with records and classical state	Measurement-assisted behavior, reset, feed-forward, normalization, and branch-sensitive cost

Table 2: VQ syntax classes and their semantic roles.

Integer and bit transformations establish reversible arithmetic before verified compilation. Measured uncomputation retains its record and branch structure. Interference arguments use the exact pure-state linear semantics.

3 Specifications preserve values, registers, cleanup, and coherence

A component specification fixes its register width, admitted basis inputs, and acceptable outputs. The output relation covers the complete basis index, so a single theorem can state the arithmetic result, preserved source fields, unchanged control bits, and zeroed workspace. Layout and placement theorems preserve a local correctness theorem when embedding the component into disjoint fields of a larger register.

Measurement-assisted components require an additional condition. For fixed immutable classical input x and admitted basis input i , let $\mathcal{B}(P, x, i)$ denote the terminal branch list of program P . The proof identifies an output index $f_x(i)$ and amplitudes $\alpha_{x,r}$ indexed by measurement records such that every $b \in \mathcal{B}(P, x, i)$ satisfies

$$|\psi_b\rangle = \alpha_{x,r(b)} |f_x(i)\rangle, \quad \sum_{b \in \mathcal{B}(P, x, i)} \|\psi_b\|^2 = 1.$$

For fixed x , the same $\alpha_{x,r}$ applies to every admitted i . By linearity, every terminal branch with record r maps an admitted finite superposition to the corresponding output superposition multiplied by $\alpha_{x,r}$. Measurement-record probabilities are therefore independent of the admitted basis input while x remains fixed.

Terminal algorithms use event probabilities over exposed outcomes. Circuit probabilities and selected-wire marginals are production functions for a computational-basis input, with exact equalities at every admitted phase

level and ordered rational bounds for the level-three model. General terminal-event probabilities for hybrid programs, including record-order and branch-splitting invariance, remain test-local.

A finite-law correctness-outside certificate binds one exact Clifford-and-CCZ point-addition program to a finite input law, an exceptional event, and a rational upper bound on its mass. It proves exact behavior for every nonexceptional case and bounds the exceptional-set mass under the stated law. Its coherence witness covers target superpositions that share one fixed immutable offset, while a quantum-state error theorem requires an additional relation between the law and the input amplitudes.

The production approximation library defines distances on finite complex vectors and matrices and a global-phase-aware terminal-vector relation. Applications supply normalization and unitarity premises. Test-local theorems derive event-probability loss bounds from operator error and compose omitted phase rotations through quantum Fourier and phase-estimation circuits. Test-local results include checked single-qubit phase-synthesis certificates and, separately, a coarse synthesized-QFT and phase-estimation construction with an accepted-weight bound. Useful-precision synthesis and a production theorem connecting generated circuits to complex operators remain open.

4 Symbolic proofs support cryptographic register widths

Direct execution reduces a finite program and gives exact results for small circuits, counterexamples, and reference calculations. Its cost grows with the materialized state dimension, the measurement branch count, and the degree of the cyclotomic ring. Cryptographic-width arithmetic uses symbolic theorems whose checking cost depends on circuit structure and arithmetic side conditions.

For reversible arithmetic, a proof follows the generator’s structure and shows how each gate changes selected bits or integer fields. Modular arithmetic adds range lemmas, congruences, and invariants, while cleanup follows from explicit reverse computation or a component-specific measured-uncomputation theorem. The compiler theorem then transfers the basis action to exact quantum semantics.

Interference proofs use finite sums and character identities. The checked QFT, phase-estimation, and elliptic-curve discrete-logarithm theorems quantify over generated widths, basis labels, phase exponents, and selected outcomes. Finite-sum lemmas reduce their terminal distributions to algebraic identities.

Resource proofs follow the recursive circuit generator. Append, reverse, and fixed repetition have production reversible gate-count certificate laws. Placement has separate production count-preservation lemmas. Algorithm-level serial repetition theorems, including the ECDLP specialization, remain test-local.

Proof method	Cost driver	Typical use
Closed exact evaluation	State dimension, branch count, and exact phase arithmetic	Small circuits, exact output amplitudes, exact probabilities, and reference cases
Basis-permutation proof	Generator length and arithmetic side conditions	Reversible adders, field arithmetic, inversion, and point addition
Finite-sum analysis	Algebraic identities and support arguments	QFT, phase estimation, terminal distributions, and recovery
Resource recurrence	Syntax recursion and numeric inequalities	Parameterized families and fixed cryptographic artifacts

Table 3: Proof methods used at different circuit scales.

5 Logical-resource theorems inspect the generated syntax

Circuit resource functions report the declared width and compute touched wires, total gates, gate classes, and dependency depth. Depth uses a greedy per-wire logical schedule in which a gate waits for its operand wires and disjoint gates may run in parallel. The functions compute exact values for the supplied circuit. Application theorems use exact values or closed upper bounds that Lean reduces and compares.

Program resources include quantum width, immutable input width, mutable classical storage, gate classes, measurements, resets, classical operations, and quantum and classical dependency depths. A branch in the syntax produces a lower and upper range by combining its arms. These endpoints may include an unreachable arm because the static analysis does not prove reachability.

Execution-cost semantics associates each terminal branch with the additive cost accumulated along its path. For in-range computational-basis inputs to a well-formed program, exact branch weights and a finite external law over those inputs define expected costs, tail masses, static upper bounds, and Markov bounds. The theory covers additive path metrics such as executed CCZ count, but defines neither expected dependency depth nor expected stopping time.

Resource certificates compose bounds for generated reversible circuits, compiled circuits, and programs formed by embedding those circuits. A resource certificate proves an inequality about syntax, while a functional theorem proves behavior. Combined application results name the same generated value in their functional and resource theorems.

Metric	Mathematical meaning	Result form	Outside the result
Quantum width	Declared whole-program register width	Exact syntax value or proved bound	Dynamic allocation and minimum-live-width analysis
Gate classes	Occurrences in the declared primitive basis	Exact syntax value or proved bound	Native hardware gates and synthesis not named by the model
Dependency depth	Greedy logical schedule over operand wires	Exact circuit value or program range	Routing, latency, factories, and physical concurrency
Hybrid operations	Measurement, reset, storage, and classical-control syntax	Exact values or branch ranges	Reachability-sensitive optimization and wall-clock timing
Expected additive cost	Semantic branch-weighted path cost under a finite input law	Exact expectation, upper bound, or tail bound	Expected depth and stochastic stopping policy
Coupling check	Each CNOT uses one graph edge, and each CCZ uses all three pairwise edges	Decidable conformance result	A verified routing transformation or native-device implementation

Table 4: Meaning and boundary of VQ resource results.

The reversible compiler represents each Toffoli by one CCZ and two Hadamards. Application reports label this count “CCZ/Toffoli-equivalent” when comparing the source arithmetic with the compiled circuit. A separate expansion charges seven T gates per CCZ. Gidney’s temporary logical AND uses four T gates for computation and none for erasure, and representing that construction requires a separate VQ semantic and resource theorem [4].

6 Project builds and the verifier check distinct theorem packages

Repository results use the project build. The kernel checks declarations against the pinned source and compiled dependencies. Axiom audits on selected result theorems report exactly proposition extensionality, classical choice, and quotient soundness.

The submission verifier checks proof packages against a restricted manifest language. Trusted code constructs each required proposition from typed manifest data, and the submitted theorem must have that exact type. The audit rejects unsafe or partial declarations, restricts reachable axioms, and invokes a separate Lean kernel replay for the submitted modules.

The claim schema determines which propositions an external package may submit. Its circuit semantic claims cover specified basis-input columns and signed permutation entries, while its probability claims cover exact certainty or impossibility and level-three rational bounds. Reversible arithmetic and selected circuit or program resource metrics have separate claim forms. Branch-indexed runtime point-addition correctness, terminal hybrid-program events, and the ECDLP semantic and probability results use project builds because the manifest language cannot express them.

The trusted boundary includes the Lean kernel, the pinned Lean and Mathlib revisions, the supporting matrix library, VQ’s definitions, and the verifier implementation that parses manifests, prepares a Lake workspace, constructs obligations, audits declarations, and drives replay. Kernel replay checks the submitted modules while

imported VQ and dependency modules remain trusted compiled artifacts. Submitted Lean source executes with the invoking user’s filesystem and network authority.

7 Applications combine semantics, probability, and resources

VQ applies its semantic and resource models to reusable arithmetic families, closed cryptographic programs, and terminal algorithm endpoints. The applications retain distinct evidence status. Their numerical bounds use the logical resource models defined in Table 4. In the ECDLP results, G denotes the standard secp256k1 generator and q denotes its subgroup order [15].

Table 5: Representative checked applications at revision `d407e8c`.

Application	Semantic, probability, or approximation result	Logical-resource result	Evidence status
Reversible arithmetic	Parameterized wrapped and modular addition, subtraction, negation, multiplication, squaring, inversion components, and placement into larger layouts	Symbolic width, gate-class, and depth formulas for selected generators	Production families and example-specific constructions
Measured addition and quantum read-only memory (QROM) cleanup	Addition and quantum-indexed lookup with input-independent measurement-record amplitudes, exact cleanup, and coherent finite-superposition action	Width, gates, measurements, resets, feed-forward, storage, and dependency-depth bounds	Production semantic interfaces and example-specific constructions
secp256k1 inverter caller	Totalized field inversion for every reduced input, returning zero on zero and the multiplicative inverse on every nonzero input, with source preservation and workspace cleanup after a proved fixed Euclidean schedule	Fixed logical width 860 and at most 2,162,792,474 CCZ/Toffoli-equivalent gates	Example-specific
Runtime secp256k1 point addition	Exact affine addition on the stated addable domain, cleanup on every representable clear-workspace input, and coherent action for each fixed runtime offset	Fixed logical width 12,317 and at most 147,228,788 CCZ/Toffoli-equivalent gates	Example-specific
Group-total point addition and scalar multiplication	The group-total mathematical translation is production. Test-local theorems cover coherent point addition and fixed-base and two-scalar circuit action on exact supported superpositions	Syntax-derived fixed-width gate-class and depth bounds	Production mathematical API and test-local circuit families
QFT, phase estimation, and order finding	Exact QFT basis action, exact grid-phase estimation, and an $N = 15$, base-two order-finding instance with exact factor probability $3/4$	Exact family formulas and closed gate-class results	Test-local and example-specific

Application	Semantic, probability, or approximation result	Logical-resource result	Evidence status
secp256k1 ECDLP	At level $\ell \geq 257$, for a valid encoded public point $Q = dG$ with $d < q$, the three-bit-window endpoint has an exact terminal distribution, accepted mass at least $((q - 1)/q)(2401/14641)$, recovery from every accepted pair, and success above 99 percent under a 26-run independent, identically distributed product law	Fixed one-run logical width 12,829, at most 25,324,153,616 CCZ/Toffoli-equivalent gates per run, and at most 658,427,994,016 across the 26-run serial unitary-gate-list record	Test-local
Approximate QFT and phase estimation	Operator-error composition, event-probability bounds, and an omitted-exact-phase ECDLP reference bound with one-run selected-event weight at least $1/32$ for the 12,831-wire circuit, level $\ell \geq 257$, a valid encoded $Q = dG$, and cutoff 17 in both QFTs	Exact formulas for omitted phase gates and the retained circuit syntax	Test-local

The runtime point-adder accepts two 256-bit affine coordinates through immutable classical input and acts on a 512-qubit affine target over secp256k1 [15]. Its value theorem uses the explicit addable domain, while its cleanup theorem covers every clear-workspace target and offset whose coordinates are below the field prime, including off-curve pairs and inputs beyond the affine value clause. The program has zero measurements, resets, and mutable classical bits, and its resource theorem concerns the same closed syntax as its correctness theorem.

The ECDLP theorem proves the terminal scalar-pair distribution and secret-recovery map for a Shor-style secp256k1 circuit [14, 12]. It derives the accepted-event lower bound from exact finite sums and analytic inequalities over every admitted public point and secret. The serial resource theorem reports the one-run circuit width and multiplies gate-list totals by 26.

The ECDLP resource result treats exact positive-sign QFT phase gates as logical primitives. Measurement, reset, classical decoding, public verification, routing, error correction, and phase synthesis with an algorithm-level error bound lie outside the stated serial gate bound. The fixed width states a whole-program register extent, and VQ has no minimum peak-live storage theorem.

Checked placement rules reuse parameterized components in larger register layouts. Artifact-level theorems state functional, probability, approximation, and logical-resource properties of one closed construction. The manifest verifier checks supplied Lean proofs against a consumer-supplied manifest within VQ’s fixed typed claim schema.

8 Existing systems cover program logic, compilation, automation, and resource inference

SQIR/VOQC and Qbricks connect circuit syntax to verified compilation or algorithm theorems, while QWIRE and CoqQ provide mixed-state semantics. Qafny reduces high-level typed programs to classical verification obligations. QuRA, Qet, and Qualtran analyze resources through type inference, symbolic expectation, or executable decomposition models.

Table 6: VQ and selected quantum verification and resource-analysis systems.

System	Program and semantics	Result	Resource and proof boundary
VQ	Lean syntax for well-formed circuits over computable exact amplitudes, reversible basis permutations, and finite hybrid programs	Functional behavior, cleanup, coherent measured components, basis-input probabilities, and test-local terminal and approximation results	Syntax-derived logical resources and basis-input expected additive costs under finite input laws. Lean checks VQ proof terms against pinned compiled dependencies. Verifier submissions and selected guarded theorems receive axiom audits [16]
SQIR/VOQC	Coq low-level circuits with unitary, density-matrix, and nondeterministic semantics	Verified optimizations, gate-set conversions, mapping, and an end-to-end Shor implementation	The Shor theorem bounds applications from SQIR’s extraction gate set. Coq checks transformations. Runnable VOQC uses extracted OCaml with a Python wrapper and trusts Coq extraction, the OCaml toolchain and runtime, and the wrapper [6, 11]
Qbricks	Qbricks-DSL circuit builders embedded in WhyML, with higher-order path-sum semantics and a deductive Hoare logic	Parameterized functional and resource specifications for QFT, phase estimation, Grover, and Shor, including exact or lower-bounded output probabilities under terminal measurement	Size, width, and ancillary-qubit bounds occur in postconditions. Why3 sends obligations to automated provers under documented axiomatic mathematical theories [1]
QWIRE	Coq typed circuits with unitary and density-matrix denotations	Circuit equations, ancilla reasoning, teleportation, and arithmetic examples	Coq checks typed construction and denotational proofs. The language has no integrated resource-certificate API. The pinned repository identifies admitted compositionality lemmas used by downstream files [10, 7]
CoqQ	Coq quantum programs over density operators and superoperators, including loops	Sound quantum Hoare logic for partial and total correctness, with algorithm case studies	Coq checks the logic soundness and applications with MathComp and MathComp Analysis. The cited logic contains no primitive-gate emission or syntax-level resource reports [18]

System	Program and semantics	Result	Resource and proof boundary
Qafny	Qafny translates high-level typed programs into classical array-like verification obligations	Sound type and proof systems with automated functional verification through Dafny	Coq establishes the metatheory. Tests validate the prototype Qafny-to-Dafny compiler. The technical report describes a separate Qafny-to-SQIR-to-OpenQASM sequence. The cited paper states no circuit-resource theorem [8]
QuRA	Proto-Quipper circuit-building programs with type-and-effect resource semantics	Parametric bounds for global circuit metrics	The tool uses type inference and satisfiability-modulo-theories (SMT) solving to infer width, gate count, T count, qubit count, and bit count. Depth appears as a paper-level local-metric example. Its soundness theorem is a paper proof [2]
Qet	Mixed classical-quantum programs over density matrices, with measurement and classical loops	Expected-cost upper bounds from explicit cost annotations	Symbolic expectation-transformer analysis reduces certificate constraints to Z3. Analyzer soundness is a paper proof [9]
Lean-QuantumAlg	Lean pure states, gates, measurement, primitives, and theorem endpoints for quantum algorithms	Lean-checked theorem endpoints for QFT, phase estimation, order finding, factoring, and discrete logarithms under stated hypotheses	The repository presents resource certificates and theorem hypotheses. It labels query and iterate counts as trusted cost annotations and cryptanalytic resources as formula-based estimates [13]
Qualtran	Hierarchical Python objects with decompositions, call graphs, and simulation interfaces	Classical and tensor simulation of constructions, with call-graph checks against full decompositions when authors supply both	Logical and surface-code physical resource estimates. Public evidence consists of executable analyses and tests [5]

Existing systems assign semantics, compilation, program logic, and resource analysis to different program representations and evidence mechanisms. Qbricks automates first-order proof obligations for circuit builders, while QuRA and Qet use solver-backed analyses for global circuit metrics and expected costs. Qualtran evaluates logical counts and physical estimates through executable decompositions. Selected VQ application theorems refer to the same Lean program in their semantic and resource statements.

9 VQ claims cover ideal finite semantics and logical resources

VQ certifies ideal finite semantics over declared logical syntax. Each theorem states the assumptions relevant to its claim, while application reports identify the input domain, gate basis, input law, and resource model where applicable. Table 7 records the remaining limits on those claims.

Table 7: Limits on VQ claims at the pinned revision.

Area	Present boundary	Consequence
Quantum states	Pure-state vectors and explicit ideal measurement branches	Density matrices, channels, physical noise, and open-system distance require separate density-operator semantics
Program syntax	Finite operation lists without loops or dynamic quantum allocation	Repetition and iterative algorithms require generated finite unrolling or a separate mathematical product model
Exact phases	Dyadic roots of unity selected by a finite semantic level	Arbitrary rotations require approximation and synthesis theorems
Approximation	Production state and operator relations. Circuit-to-complex execution, event-probability bounds, and phase-synthesis theorems remain test-local	Production integration of phase-synthesis error with terminal success remains open
Evaluation	Direct state and branch evaluation is exponential	Cryptographic-width proofs require handwritten symbolic semantics, invariants, sums, and resource recurrences
Compilation	Verified reversible-to-unitary compilation	General optimization, external assembly translation, and routing transformations remain unverified
Public API	Terminal hybrid observation, ECDLP, QFT, phase estimation, repetition, approximation transport, and phase synthesis remain under the test root	Their theorems are kernel checked but do not yet define a reviewed production interface or manifest claim
Submission checking	Submitted Lean source runs with the invoking user’s authority	The verifier’s theorem claim assumes trusted execution of submission source

At revision `d407e8c`, VQ certifies exact ideal behavior or stated logical-resource bounds for artifacts covered by its theorems within explicit domains. Across separate applications, the repository establishes value semantics, preservation, cleanup, coherent measured execution, analytic probabilities, and syntax-derived resources. Claims about physical execution require additional verified models for synthesis, routing, fault tolerance, noise, and timing.

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