

Lean Verification of Schrottenloher’s Optimized Elliptic-Curve Discrete-Logarithm Circuits: A Progress Report

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Abstract

Schrottenloher proposed low-resource circuits for the elliptic-curve discrete logarithm problem (ECDLP) that combine a compressed binary Euclidean trace, reversed Bézout reconstruction, and approximate modular arithmetic. This report describes a Lean 4 formalization in VQ at artifact revision `d090d5a`. The checked development proves an exact $2n$ -round termination bound, reversible trace compression, complete basis-state semantics and cleanup for Algorithms 2–4, a secp256k1 instance, and syntax-derived logical resource bounds. Separate theorems specify success domains for Algorithms 6–7 and value correctness for Algorithms 9–11, while the Algorithm 1 work covers its affine value trace and first physical lookup–subtraction phase. The complete point-addition circuit, the probability of approximate-arithmetic failure, and the paper’s optimized resource totals remain unverified.

1 The artifact separates three kinds of evidence

Schrottenloher’s construction reduces windowed elliptic-curve point addition to modular arithmetic, with in-place multiplication as its largest reported subcomponent [7]. Algorithms 2 and 3 first record a binary Euclidean execution and then consume that record in reverse to update a pair of modular coefficients. Algorithm 4 composes the forward trace, coefficient reconstruction, and reverse trace to implement $|x, y\rangle \mapsto |x, xy \bmod q\rangle$ for nonzero x .

The VQ development distinguishes a *value model*, a *circuit semantics theorem*, and a *resource theorem*. A value model is a total Lean function over natural numbers, Booleans, or finite records. A circuit semantics theorem proves the action of generated reversible gate syntax on a packed basis index, including the contents of work fields. A resource theorem counts or bounds that same syntax. It does not inherit a number copied from the paper. These distinctions follow VQ’s existing semantics and certification method [9, 8].

Table 1 records the current paper-specific boundary. “Value” means that Lean proves the arithmetic equation on a stated domain, while “circuit” additionally means that the generated gate list has a checked whole-index action and cleanup equation. A partial resource result covers only the named checked component.

2 Algorithms 2–4 have an exact circuit theorem

The arithmetic proof starts from a pair (u, v) with odd u . One Algorithm 2 step records whether v is odd and whether an odd v is smaller than u , then performs the corresponding swap, subtraction, and division by two. Lean proves that the left component remains odd, coprimality persists, and the product satisfies

$$2u'v' \leq uv$$

for every live step. Iteration gives $2^k u_k v_k \leq u_0 v_0$, which implies that coprime inputs below 2^n reach $(1, 0)$ after $2n$ steps. The endpoint is a fixed point, so padding the schedule preserves it.

Each trace choice belongs to the three-element set $(0, 0), (1, 0), (1, 1)$. Three choices therefore have 27 valid values and fit in five bits. The verified Figure 1 circuit maps each valid six-bit input to a five-bit retained value and a cleared sixth bit, recovers the original block when reversed, and uses 5 Toffoli, 6 CNOT, and 2 X gates.

Paper component	Value semantics	Circuit semantics	Resource status
Algorithm 1	Affine branch and $i = 0$ branch	First lookup, two coordinate subtractions, and inverse lookup	Reference selector and first phase only
Algorithms 2–4 and Figure 1	Complete for an exact $2n$ schedule	Complete forward–reconstruct–reverse equation	Closed secp256k1 bounds
Algorithms 5 and 8	Equivalent exact modular operations	Reusable exact components, without pseudocode refinement	Component formulas
Algorithms 6 and 7	Correctness on explicit success domains	Open	Open
Algorithms 9–11	Correctness on explicit success domains	Open	Open
Windowed ECDLP circuit	Existing group and Fourier results outside this paper-specific refinement	Open	Open

Table 1: Paper-specific theorem coverage at revision d090d5a.

The family proof handles full blocks and a final block containing one or two genuine choices, without applying artificial padding choices.

The reconstruction invariant associates a coefficient pair (r, s) with each remainder pair. Reversing one stored choice doubles s modulo q , adds r when the odd bit is set, and conditionally swaps the coefficients. Lean proves that this update reverses the corresponding Algorithm 2 step and preserves the required modular linear relation. A descending block induction decompresses one block, consumes its genuine choices in reverse, clears its six working bits, and restores the preceding Algorithm 2 state.

The composed circuit theorem quantifies over every odd prime $q < 2^n$, every $0 < x < q$, every $y < q$, and a modular-adder circuit satisfying the checked **AddsMod** specification. On the canonical trace and clean coefficient workspace it restores the complete trace input and maps coefficient fields $(y, 0)$ to $(0, xy \bmod q)$. The secp256k1 specialization discharges primality, width, adder semantics, and well-formedness at $n = 256$ and 512 rounds [8].

3 The clean in-place interface completes Algorithm 4

Revision d090d5a contains the verified schedule, its closed secp256k1 specialization, and the external transformation required by Algorithm 4. The clean wrapper copies x into a trace field, loads q , runs the verified schedule, moves the product into the original y field, and reverses the preparation. Its checked equation preserves x , replaces y by $xy \bmod q$, and restores one combined work field to zero.

The same module proves the reverse-circuit equation. For nonzero $x < q$, reversing the circuit maps y to $yx^{-1} \bmod q$, using the checked prime-field inverse law. The secp256k1 instance has width 2,401, compared with 2,145 wires for the inner trace-and-coefficient circuit. Focused validation job 98 and aggregate job 99 returned stored status zero before commit d090d5a, so every Lean claim in this report belongs to the cited committed revision. The report makes no claim from an uncommitted Lean source.

4 Resource theorems cover the exact reference circuit

The parameterized resource proof follows the same generated syntax as the semantic theorem. If $B = \lceil \text{rounds}/3 \rceil$ and one coefficient update has raw, CCX, and CX counts K_ℓ, K_t, K_c , the complete inner circuit has

$$\begin{aligned}
G_{\text{raw}} &= \text{rounds}(K_\ell + 80n + 16) + 26B, \\
G_{\text{CCX}} &= \text{rounds}(K_t + 40n + 2) + 10B, \\
G_{\text{CX}} &= \text{rounds}(K_c + 40n + 10) + 12B.
\end{aligned}$$

These formulas count the complete forward and reverse schedule. Its width is $4n + 5B + 5 + \max(n + 5, w_a + 1)$. The compiled primitive count follows from these equations, while total and Toffoli depth currently use gate-count upper bounds.

The closed `secp256k1` instance has width 2,145. Its upper bounds are 19,687,774 raw reversible gates, 12,593,326 compiled Toffoli gates, 10,236,420 CNOT gates, and 44,874,426 compiled primitive gates. These figures describe the exact 512-round reference implementation, including reversible uncomputation. They cannot be compared as reproductions of Schrottenloher’s Tables 1–3: the paper uses a shorter empirical schedule, truncated register sharing, approximate arithmetic, measurement-based uncomputation, and a “Toffoli” metric that groups several operations. The checked selector lookup likewise gives a semantic reference whose 4,194,304-Toffoli upper bound at $w = 16$ is 64 times the paper’s optimized 2^{16} lookup count [8].

5 Approximate arithmetic now has explicit success predicates

The Algorithm 6 value model retains the paper’s most-significant-bit comparison. If $H_k(z)$ denotes the retained high part, Lean proves correctness when

$$q \leq 2x \quad \text{or} \quad H_k(2x) < H_k(q).$$

Failure therefore requires $2x < q$ and $H_k(2x) = H_k(q)$, which makes the comparison request a false subtraction. The Algorithm 7 theorem proves the pseudo-Mersenne result when doubling stays below q , or when an overflow correction by $f = 2^n - q$ produces no carry beyond the selected low field. Its failure theorem separates a missed reduction in $[q, 2^n)$ from a discarded low-field carry.

Algorithms 9–11 have total value models and exact correctness theorems on analogous success domains. The Algorithm 9 model treats the sum in an $(n + 1)$ -bit register. The Algorithm 10 model states the overflow and truncated f -addition conditions, while the Algorithm 11 model adds the special $x + y = q$ branch. These results prove arithmetic values, but they do not yet prove the pseudocode circuits’ flag erasure, work-field cleanup, coherent control behavior, or failure probability [8].

The paper reports a failure probability at most $2^{-13.3}$ and says that its circuits succeeded on 10,000 random inputs [7]. A sample with no observed failures does not establish that probability upper bound without a stated sampling model and statistical or deductive argument. The Lean success predicates identify the events whose probability a later proof must bound, but the current development supplies no distribution theorem for those events and no composition bound for a complete point addition.

6 Algorithm 1 exposes its affine-domain conditions

The Algorithm 1 value model follows the paper’s arithmetic order after the first lookup. For the enabled branch, Lean proves equality with affine point addition under the distinct-input and recoverable-output hypotheses required by the divisions. For the $i = 0$ branch, where the lookup returns $(0, 0)$, it proves that the accumulator remains unchanged under the current sentinel and field-inverse assumptions. The paper states that its formulas neglect doubling and the point at infinity except for the $i = 0$ case, so the formal hypotheses make an acknowledged scope restriction precise rather than expose a contradiction.

The first physical phase accepts any coherent XOR lookup and any verified modular subtractor. It loads the selected 512-bit affine point, subtracts both coordinates from the two 256-bit target fields, and reverses the lookup. The whole-index theorem restores the lookup output and both work fields and changes only the target coordinates. The next physical obligation is conditional slope division, followed by the second lookup, controlled squaring, the clean in-place multiplier, and the final lookup and arithmetic sequence.

The principal source endpoints make these claims reproducible. The complete schedule and resource formulas reside in `SchrottenloherRawChoiceSchedule`, while `VQBridge.Euclid.Schrottenloher` instantiates the clean `secp256k1` forward and reverse equations. The approximate-arithmetic and Algorithm 1 test modules contain the remaining paper-specific endpoints at the cited artifact revision [8].

7 Kernel dependencies and reproducibility

The checked public endpoints have permanent `#print axioms` guards. Lean reports a subset of three standard dependencies: `propext`, `Classical.choice`, and `Quot.sound`. The arithmetic termination chain avoids

`Classical.choice` until the prime specialization invokes `mathlib`’s primality-to-coprimality theorem. These are standard Lean or `mathlib` dependencies rather than project-defined axioms [2, 10, 5]. The audited modules contain no `sorry`, `admit`, project axiom, `native_decide`, or source-level recursion or heartbeat override.

Revision `d090d5a` is the committed evidence boundary for the results described in Table 1. The repository journal records focused and aggregate checks and preserves the distinction between successful Lean jobs, failed source-debugging attempts, and interrupted infrastructure runs. The report cites the immutable repository revision rather than treating the prose journal as proof evidence. The Lean declarations and their guarded tests constitute the checkable artifact.

8 The mechanization contributes refinement evidence

Schrottenloher’s paper and implementation provide the circuit architecture, pseudocode, simulations, and estimated resources [7]. The cited antecedents provide the dialog representation, exact modular components, QROM, and elliptic-curve circuit designs on which that architecture builds [4, 6, 1, 3]. Schrottenloher’s paper cites no theorem-prover artifact for its Algorithms 1–11.

The contribution of this development is the machine-checked refinement evidence now available for the exact core: a deterministic $2n$ termination bound, the physical five-bit compression family, reverse block consumption with partial-final-block cleanup, the complete Algorithms 2–4 basis-state equation, and syntax-derived resource formulas. The approximate-arithmetic models add exact success predicates and failure characterizations to the paper’s empirical account. This novelty claim concerns the formal statements and their composition. The algorithms and optimization ideas belong to the cited work.

The remaining theorem boundary is specific. A complete result needs the Algorithm 1 circuit after the first subtraction phase, circuit refinements for Algorithms 6–7 and 9–11, probability bounds for their failure events under the ECDLP input state, a checked measurement-based $w = 16$ unlookup, and one composition theorem connecting point multiplication to Fourier measurement and classical recovery. Until those results exist, the artifact verifies an exact reference multiplier and selected point-addition components rather than the paper’s complete low-resource ECDLP circuit.

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